1. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+2}}\,\mathrm{d}x$.

   Solution:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2}}\,dx &= \int \frac{\sqrt{2}\sec^2\theta}{\sqrt{2+2\tan^2\theta}}\,d\theta \quad (x=\sqrt{2}\tan\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}; dx=\sqrt{2}\sec^2\theta\,d\theta)\\ &= \int \sec\theta\,d\theta + C\\ &= \ln|\sec\theta+\tan\theta| + C\\ &= \ln\left|\frac{\sqrt{x^2+2}}{\sqrt{2}} + \frac{x}{\sqrt{2}}\right| + C\\ &= \ln|\sqrt{x^2+2}+x| + C_1\end{aligned}$$

2. Evaluate $\displaystyle\int \frac{16}{x^3-4x}\,\mathrm{d}x$.

   Solution:

$$\begin{aligned}\int \frac{16}{x^3-4x}\,dx &= \int \left(\frac{-4}{x}+\frac{2}{x+2}+\frac{2}{x-2}\right)dx\\ &= -4\ln|x| + 2\ln|x+2| + 2\ln|x-2| + C\end{aligned}$$

3. The region under the graph of $y=\sin x$, $0\le x\le\pi$, is rotated 360 degrees about the y-axis forming a solid of revolution S. Find the volume of S.

   Solution: Via the method of cylindrical shells, we obtain the following integral yielding the volume of S:

$$2\pi\int_0^\pi x\sin(x)\,dx.$$

   Integrating by parts with $u=x, dv=\sin x\,dx, du=dx$, and $v=-\cos x$, we obtain

$$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

   Thus, the volume of S is

$$2\pi\int_0^\pi x\sin(x)\,dx = 2\pi\left[-x\cos x+\sin x\right]_0^\pi = 2\pi^2.$$

4. Solve the differential equation, obtaining an explicit solution:

$$\frac{dy}{dx} = y^2x^2 + y^2.$$

   Solution: Separating variables and integrating, we obtain

$$\int \frac{1}{y^2}\,dy = \int (1+x^2)\,dx,$$

   which yields an implicit general solution

$$-\frac{1}{y} = x + \frac{x^3}{3} + C.$$

Solving for y, we obtain an explicit general solution

$$y = -\frac{1}{x + \frac{x^3}{3} + C}.$$

5. Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges.

Solution: The series may be written $\sum_{n=2}^{\infty} f(n)$, where $f(x) = \frac{1}{x(\ln x)^2}$. Note

$$\begin{aligned}
\int_2^{\infty} f(x)\,dx &= \lim_{t\to\infty} \int_2^t (\ln x)^{-2} \frac{1}{x}\,dx \\
&= \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_2^t \\
&= \lim_{t\to\infty} \left(\frac{-1}{\ln t} + \frac{1}{\ln 2}\right) \\
&= \frac{1}{\ln 2}.
\end{aligned}$$

Since f is continuous, positive, and decreasing on $[2, \infty)$, and, by the preceding computation, the improper integral $\int_2^{\infty} f(x)\,dx$ converges, we conclude the infinite series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by the Integral Test. *Remark: When grading this problem, consider your solution "correct" provided you found the value of the improper integral* $\int_2^{\infty} f(x)\,dx$ *to be* $1/(\ln 2)$ *and concluded that the series converges by the Integral Test.*

6. Find the Maclaurin Series for $f(x) = \dfrac{x}{3 + x^2}$. What is the radius of convergence?

Solution:

$$\begin{aligned}
f(x) &= \frac{x}{3 + x^2} \\
&= \frac{x}{3} \frac{1}{1 + \frac{x^2}{3}} \\
&= \frac{x}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^2}{3}\right)^n \qquad \left(\text{for } x \text{ satsifying } \left|\frac{x^2}{3}\right| < 1\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^{2n+1}
\end{aligned}$$

Thus, the Maclaurin series of f is $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^{2n+1}$ and, since it converges for precisely those x satisfying

$$|x| < \sqrt{3},$$

its radius of convergence is $\sqrt{3}$.

7. Determine the radius and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-1)^n$.

Solution: We employ the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)2^{n+1}}(x-1)^{n+1}}{\frac{1}{n2^n}(x-1)^n} \right| &= |x-1| \lim_{n\to\infty} \frac{n}{2(n+1)} \\
&= |x-1| \cdot \frac{1}{2}.
\end{aligned}$$

Thus, the series converges (absolutely) for those x satisfying $\frac{1}{2}|x-1| < 1$, that is, for $-1 < x < 3$. It diverges for $|x-1| > 2$. Thus, the radius of convergence of the power series is 2 and its interval of convergence has endpoints -1 and 3. At the left endpoint $x = -1$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}(-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating-Series Test (because the sequence $(1/n)$ decreases to 0). At the right endpoint $x = 3$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}(2)^n = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is the divergent harmonic series. Hence, the interval of convergence of the given power series is $[-1, 3)$.

8. Suppose that (x, y) has distance $r \geq 1$ from the origin. Use polar coordinates to show that

$$(|x| + |y|)\ln(x^2 + y^2) \leq 4r\ln(r).$$

Solution: Suppose that (x, y) has distance $r \geq 1$ from the origin. Choose θ such that $x = r\cos\theta$ and $y = r\sin\theta$; then, we have

$$\begin{aligned}
(|x| + |y|)\ln(x^2 + y^2) &= (|r\cos\theta| + |r\sin\theta|)\ln\left(r^2\left(\cos^2(\theta) + \sin^2(\theta)\right)\right) \\
&= (|r||\cos\theta| + |r||\sin\theta|)\ln\left(r^2\right) \\
&\leq (|r| + |r|)\ln(r^2) \qquad (\ln(r^2) \geq 0 \text{ because } r^2 \geq 1) \\
&= 4|r|\ln(r) \qquad (\text{laws of logarithms}).
\end{aligned}$$

9. Find an equation of the line tangent to the curve $x = 1 + \ln t$, $y = t^2 + 2$ at the point $(1, 3)$.

Solution: The slope of the tangent line is given by $\frac{y'(1)}{x'(1)} = \frac{2}{1}$, and thus, an equation of the tangent line is $y - 3 = 2(x-1)$. Alternatively, one might observe that $t = e^{x-1}$, so that $y = e^{2x-2} + 2$. Thus, $\frac{dy}{dx} = 2e^{2x-2}$, yielding the tangent-line slope $\left.\frac{dy}{dx}\right|_{x=1} = 2$.

10. Find the length of the curve $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 2$.

Solution: The length of the curve is given by

$$\begin{aligned}
\int_0^2 \sqrt{x'(t)^2 + y'(t)^2}\,dt &= \int_0^2 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\,dt \\
&= \int_0^2 \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\,dt \\
&= \int_0^2 \sqrt{(e^t + 1)^2}\,dt \\
&= \int_0^2 (e^t + 1)\,dt \\
&= \left[e^t + t\right]_0^2 \\
&= e^2 + 2 - 1 \\
&= e^2 + 1.
\end{aligned}$$