Nick Hamblet – Poset of linear subspaces in $\mathbb{R}^k$

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Let R^k denote the poset of proper, non-trivial, linear subspaces of $\mathbb{R}^k$.

Ex $\quad k=2, \quad |R^2| = \mathbb{R}P^1 \simeq S^1$

$k=3, \quad R^3: \quad \left| G_1(\mathbb{R}^3) \xrightarrow[\text{Flag}(\mathbb{R}^3)]{} G_2(\mathbb{R}^3) \right| \quad =$ push-out

$$\begin{tikzcd}
\text{Flag} \arrow[r] \arrow[d] & GL_1(\mathbb{R}^3) \arrow[d] \\
GL_2(\mathbb{R}^3) \arrow[r] & |R^3|
\end{tikzcd}$$

Goal: $\quad |R^k| \simeq S^{\frac{k(k+1)}{2} - 2}$

Why look at this?

$$\Sigma^\infty C(k,V) \simeq \underset{\text{poset}}{\text{holim}}\ V_m S$$

$\uparrow$ configuration space

Can analyse using orthogonal calculus.

$$\Sigma^\infty \underset{\uparrow \text{injective}}{\text{Mor}}(\mathbb{R}^n, V) \simeq \underset{0 \neq E \leq \mathbb{R}^n}{\text{holim}} \left(\hom(\mathbb{R}^n, V) - \hom((\mathbb{R}^n, E), (V, 0)) \right)$$

$$\simeq \text{Nat}(|\mathbb{R}^n \downarrow -|, F) \qquad \left(\text{holim}_{\mathcal{C}} F \simeq \text{Nat}(|\mathcal{C} \downarrow -|, F(-)) \right).$$

R^k is a category internal in $\mathcal{T}op$:

Space of $\text{Obj} = \coprod_{i=1}^{k-1} Gr(i)$

Space of $\text{Mor} = \coprod_{1 \leq i \leq j \leq k-1} \text{Flag}(i \leq j)$

$$\begin{array}{c}
\text{Mor} \times_{\text{Obj}} \text{Mor} \\
\downarrow \uparrow \downarrow \uparrow \downarrow \\
\text{Mor} \\
s \downarrow \ \uparrow \text{id} \ \downarrow t \\
\text{Obj}
\end{array}$$

Form the simplicial nerve $\mathcal{N}_\bullet R^k$ has $\mathcal{N}_\ell R^k = \coprod (\text{Flags of length } \ell + 1)$

$$|\mathcal{N}_\bullet R^k| =: |R^k|$$

Let $\text{Symm}(k)$ denote the space of $k \times k$ real symmetric matrices. $\left(\dim \text{Symm}(k) = \frac{k(k+1)}{2}\right)$

Define $\quad N_0 R^k \xrightarrow{f} \text{Symm}(k)$

$$E \longmapsto \begin{array}{c c} & \begin{array}{cc} E & E^\perp \end{array} \\ \begin{array}{c} E \\ E^\perp \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

Extend f to a map

$|\mathcal{N}_\bullet R^k| \longrightarrow \text{Symm}(k) \qquad$ linearly:

For $p \in |R^k|$, suppose p is in the $d-1$ simplex corresponding to the flag

$$0 < E_1 < \cdots < E_d < \mathbb{R}^k$$

w/ coordinates $0 < \tau_1 < \cdots < \tau_{d-1} < 1$

Let $\tau_0 = 0$, $\tau_d = 1$, Then $f(p)$ is the map that is multiplication by

Let $F_i = E_i \cap E_{i-1}^\perp$ τ_i on F_i

$F_0 = E_d^\perp$

Include $\mathbb{R}$ into $\mathrm{Symm}(k)$ as the "constant diagonal" matrices

This acts on $\mathrm{Symm}(k)$ by addition, giving

$\mathrm{Symm}(k)/\mathbb{R}$

There is a nice representative for each equivalence class: 0 is the smallest eigenvalue.

Since $0 \in \mathrm{Symm}(k)/\mathbb{R}$ has only one eigenvalue, $f(p) \neq 0$, so descends to a map

$$|R^k| \longrightarrow \left(\mathrm{Symm}(k)/\mathbb{R} - \{0\}\right)/\mathbb{R}_{>0} \cong S^{\frac{k(k+1)}{2}-2}$$

representatives of classes are all eigenvalues w/ least 0 and greatest 1.

Then it is visibly a bijection. The target is Hausdorff, the source compact $\Rightarrow$ homeomorphism.

So for example, $|R^3| = S^4$.

Remark: Given a semisimple Lie group, have a unique maximal compact, the quotient by which is a negatively curved symmetric space. Looking at flags of flat things embedded therein. Gives essentially the same idea.