Well-posedness for the Navier-Stokes equations

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We study the incompressible Navier-Stokes equations on $\mathbb{R}^n \times \mathbb{R}^+$

$$u_t + (u \cdot \nabla)u - \Delta u + \nabla p = 0 \qquad \nabla \cdot u = 0 \tag{1}$$

where u is the velocity and p is the pressure, with inital data $u(x,0) = u_0(x)$.

Existence of weak solutions has been shown by Leray. Uniqueness (and regularity) of weak solutions is unknown and both are among the major open questions in applied analysis. Under stronger assumptions there exist local and/or global smooth solutions. One version of this has been shown by Kato for initial data in $L^n(\mathbb{R}^n)$.

What are reasonable requirements for solutions and initial data?

I. The nonlinearity is quadratic. To have the notion of a weak solution we have to require for all x and R

$$u \in L^2(B_R(x) \times [0, R^2]) \tag{2}$$

II. The Navier-Stokes equations are invariant with respect to scaling. If u is a solution (we omit the pressure in the notation, as well as a discussion what we mean by a solution), then

$$\tilde{u} = \lambda u(\lambda x, \lambda^2 t)$$

is again a solution. We require the scale invariant version of the L^2 condition above

$$\sup_{x, t \le T} t^{-n/2} \int_{B_{\sqrt{t}}(x)} \int_0^t |u|^2 \, dy \, ds < \infty. \tag{3}$$

III. The linear term should be dominant. Let

$$\|v\|_{BMO_T^{-1}} := \sup_{x, t \le T} \left(t^{-n/2} \int_{B_{\sqrt{t}}(x)} \int_0^t |u|^2 \, dx \, ds \right)^{1/2} \tag{4}$$

where u is the unique function which satisfies

$$u_t - \Delta u = 0, \qquad u(x,0) = v(x).$$

We define the function space BMO_T^{-1} as the set of all tempered distributions for which the norm is finite.

Theorem 1. *There exists $\delta > 0$ with the following property: Let $0 < T \le \infty$. If $\nabla \cdot u_0 = 0$ and $\|u_0\|_{(BMO_T^{-1})^n} \le \delta$ then there exists a unique smooth solution up to time T.*

The theorem implies for example existence of a unique local solution for initial velocity in L^n or L^∞.

The talk is based on joint work with D. Tataru.

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