Topology General Exam **August 15, 2023**

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style on your own paper. All work should be your own—no outside sources are permitted—using methods and results from the first year topology courses. Each problem is worth the same number of points.

1. Let x_i, y_i, z, $i = 1, \ldots, n$, denote the coordinates on $\mathbb{R}^{2n+1}$. Consider the 1-form

$$w = \mathrm{d}z + \sum_{i=1}^{n} x_i \, \mathrm{d}y_i.$$

Compute $w \wedge (\mathrm{d}w)^n$ and show that it is a volume form. Here $(\mathrm{d}w)^n$ denotes the n-fold wedge product $\mathrm{d}w \wedge \ldots \wedge \mathrm{d}w$.

2. Consider the subset S of $\mathbb{R}^4$ defined by the equation $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1$, and consider the function $f \colon \mathbb{R}^4 \longrightarrow \mathbb{R}$ given by $f(x_1, x_2, x_3, x_4) = x_1$.

Show that S is a submanifold of $\mathbb{R}^4$, and identify the critical points and critical values of the function f restricted to S, $f|_S \colon S \longrightarrow \mathbb{R}$.

3. Let M be a compact manifold with boundary. Prove that there does not exist a smooth map $f \colon M \longrightarrow \partial M$ such that $f(x) = x$ for all $x \in \partial M$.

4. Let M be a smooth manifold and $f \colon M \longrightarrow M$ a smooth map. Suppose that $p \in M$ is a fixed point of f, meaning $f(p) = p$. Prove that the following conditions are equivalent:

a) The graph $\{(x, f(x)) | x \in M\}$ of f intersects the diagonal $\{(x, x) | x \in M\}$ transversely at the point $(p, p) \in M \times M$.
b) 1 is not an eigenvalue of the differential $d_p f \colon T_pM \longrightarrow T_pM$.

Hint. It may be helpful to consider the graph $\{(v, d_pf(v)) | v \in T_pM\}$ of the differential d_pf and the conditions under which it is transverse to the diagonal $\{(v, v) | v \in T_pM\}$ in $T_pM \times T_pM$.

5. Suppose that the 3-torus $T^3 = S^1 \times S^1 \times S^1$ is written as a union $T^3 = U_1 \cup \cdots \cup U_m$ of open subets U_j each homeomorphic to $\mathbb{R}^3$. Prove that for some $k \leq m$, the intersection $(U_1 \cup \cdots \cup U_{k-1}) \cap U_k$ is either empty or disconnected.

6. Let $f : S^n \to S^n$ be a continuous map.

a) Prove that if $\deg(f) \neq (-1)^{n+1}$ then f has a fixed point.
b) Suppose that $f(x) = f(-x)$ for all $x \in S^n$. Prove that $\deg(f)$ is even, and if in addition n is even, then $\deg(f) = 0$.

7. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$.

a) Prove that X does not admit a connected, regular, 3-sheeted covering space.
b) Construct a connected (non-regular) 3-sheeted covering space of X.

8. Let $K \subset S^3$ be a smooth submanifold diffeomorphic to a circle, $K \cong S^1$ (so K is a "knot" in the 3-dimensional sphere). Use the axioms and properties of homology, including for example the long exact sequence of a pair and the Mayer-Vietoris sequence, to answer the following:

a) Find the relative homology groups $H_j(S^3, K)$ for all j.

b) Prove that $H_1(S^3 - K) \cong \mathbb{Z}$.