1. Recall that if X is a "nice" (meaning connected, locally path connected and semi-locally simply connected) topological space with basepoint x_0, then connected, based covering spaces of X are in 1–1 correspondence with subgroups of $\pi_1(X, x_0)$. For such a space X, let $p : \widetilde{X} \to X$ be a universal covering map, and let $A \subset X$ be a "nice" subspace. Let $\widetilde{A} \subset \widetilde{X}$ be a path component of the preimage $p^{-1}(A)$. Prove that, with suitable choices of basepoints, the restriction $p : \widetilde{A} \to A$ is the covering space of A corresponding to the kernel of the map $\pi_1(A) \to \pi_1(X)$ induced by inclusion.

2. a) For each $n > 0$, let S^n have the CW structure with a single 0-cell and a single n-cell. Describe a CW structure on the product $S^n \times S^m$, and use it to compute the homology groups $H_i(S^n \times S^m)$.

b) Let X be the space obtained from $S^1 \times S^2$ by attaching a 2-cell along a map $\partial D^2 \to S^1 \times \{pt\}$ of degree k. Describe a CW structure on X and use it to find the homology groups of X.

3. Let $f : S^n \to S^n$ be a continuous map that is homotopic to a constant map. Prove that there exists $x \in S^n$ with $f(x) = x$, and also a point $y \in S^n$ with $f(y) = -y$.

4. Let M be a smooth, oriented manifold without boundary and $f : M \to \mathbb{R}$ a smooth map. For $t \in \mathbb{R}$ a regular value of f, prove that the submanifold $Y = f^{-1}(t)$ of M is orientable.

5. Let M be a smooth manifold covered by two open connected sets, $M = U_1 \cup U_2$. Suppose ω is a 1-form on M and $f_i : U_i \to \mathbb{R}$ are smooth maps such that $\omega_{U_i} = df_i$, $i = 1, 2$ Prove that if $U_1 \cap U_2$ is connected then there is a smooth function $f : M \to \mathbb{R}$ such that $\omega = df$. Find an example of a manifold $M = U_1 \cup U_2$ where $U_1 \cap U_2$ is not connected, and this conclusion does not hold.

6. Prove that the orthogonal group $O(n)$ (consisting of $n \times n$ real matrices whose rows are orthonormal) is a manifold. What is its dimension?

7. Let A and B each be copies of a Möbius band, and let $X = A \cup B$ be the space obtained by identifying the boundary circles of A and B.

a) Find a presentation for the fundamental group of X.
b) Compute the homology groups of X with $\mathbb{Z}$ coefficients and with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

8. Consider the Möbius band M with boundary curve γ. Denote the two circles in the wedge sum $S^1 \vee S^1$ by a, b, and consider a continuous map $\partial M \longrightarrow S^1 \vee S^1$ which sends γ to the word ab^2ab. Does f admit an extension to M?

A possible alternative to one of the above problems:

9. Let $f\colon A \longrightarrow B$ be a chain map. Denoting the differential in the chain complexes A, B by $\mathrm{d}^A, \mathrm{d}^B$ respectively, the *mapping cone* $C(f)$ is defined as the chain complex

$$\ldots \longrightarrow A_{n+1} \oplus B_n \longrightarrow A_n \oplus B_{n-1} \longrightarrow A_{n-1} \oplus B_{n-2} \longrightarrow \ldots$$

with the differential

$$\begin{pmatrix} -d^A_{n+1} & 0 \\ f_{n+1} & d^B b_n \end{pmatrix}.$$

Prove that if f, g are chain homotopic, then $C(f), C(g)$ are chain homotopy equivalent.