Topology General Exam **August 15, 2022**

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style on your own paper. All work should be your own—no outside sources are permitted—using methods and results from the first year topology course topics. Each problem is worth the same number of points.

1. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve. Prove that the set of real numbers

$$K = \{r > 0 \,|\, \text{the sphere of radius } r \text{ around } 0 \in \mathbb{R}^n \text{ is tangent to the image of } \gamma\}$$

has measure zero in $\mathbb{R}$.

2. (a) Let M and N be connected compact smooth manifolds of the same dimension (without boundary), and $f : M \to N$ a submersion. Prove that f is a covering map.

(b) Suppose M is a connected closed surface (2-manifold), and $f : M \to S^2$ a submersion. Prove that f must, in fact, be a diffeomorphism.

3. Consider the subset S of $\mathbb{R}^4$ defined by the two equations

$$x^2 + y^2 - z^2 - w^2 = 1$$
$$xy + z + w = 3.$$

Prove that S is a smooth manifold and find its dimension.

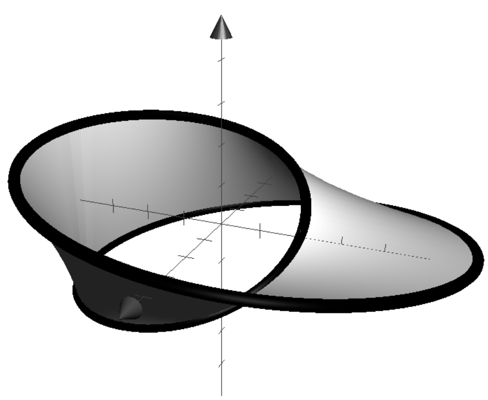
4. Consider the smooth, closed curve γ in $\mathbb{R}^3$ given in cylindrical coordinates[1] by

$$\gamma(t) = (r(t), \theta(t), z(t)), \qquad \begin{aligned} r(t) &= 2 + \sin t \\ \theta(t) &= 2t \\ z(t) &= \cos t \end{aligned}$$

where $t \in [0, 2\pi]$. Visually, γ can be pictured as the boundary of a Möbius band $M \subset \mathbb{R}^3 - \{r = 0\}$ pictured below. Let α be the 1-form on $\mathbb{R}^3 - \{r = 0\}$ defined by $\alpha = d\theta$.

a) Find $\int_\gamma \alpha$.

b) Use your answer to (a) to prove that Möbius band M is not orientable, and in fact γ is not the boundary of any smooth, compact, orientable surface in $\mathbb{R}^3 - \{r = 0\}$.



[1] Recall that r is the distance from the z-axis, and θ is the usual angle about the z-axis.

5. (a) Show that if $f : S^4 \to S^4$ is continuous, then $f \circ f : S^4 \to S^4$ must have a fixed point.

(b) By contrast, show that there is a continuous map $f : S^3 \to S^3$ such that $f \circ f$ has no fixed point.

6. Let X_n denote the n–skeleton of a CW complex X.

a) Complete the definition: The **cellular chain complex** (C_*^{CW}, d_*) **of** X is defined by letting $C_n^{CW}(X) = H_n(X_n, X_{n-1})$ and then letting $d_n : C_{n+1}^{CW}(X) \to C_n^{CW}(X)$ be *[you complete the definition]*.

b) Using your definition from part (a), show that $d_{n-1} \circ d_n = 0$ for all $n \geq 1$.

c) Prove that $C_n^{CW}(X)$ is isomorphic to a free abelian group with one generator for each n–cell of X.

7. Let $S^3 \xrightarrow{i_1} S^3 \vee S^3 \xleftarrow{i_2} S^3$ be the two inclusion maps into the wedge. Say that a map $f : S^3 \vee S^3 \to S^3$ has type (m, n) if the degree of $f \circ i_1$ is m and the degree of $f \circ i_2$ is n. Let $X_f = S^3 \cup_f (D^4 \vee D^4)$, i.e. X_f is the pushout

$$\begin{tikzcd} S^3 \vee S^3 \arrow[r, "f"] \arrow[d] & S^3 \arrow[d] \\ D^4 \vee D^4 \arrow[r] & X_f. \end{tikzcd}$$

Compute the homology groups of X_f if f has type $(9, 6)$, describing the homology groups as direct sums of cyclic groups, as usual.

8. A group G is *perfect* if its commutator subgroup[2] $[G, G]$ is all of G. The famous Poincaré 'sphere' is a 3–dimensional manifold M whose fundamental group is a perfect group of order 120. Show that any continuous function $M \to \mathbb{R}P^4$ must be null homotopic.

(You may assume that f is smooth if desired.)

[2]The subgroup generated by elements of the form $ghg^{-1}h^{-1}$.