Topology General Exam **January 10, 2022**

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style. All work should be your own—no outside sources are permitted—using methods and results from the first year topology course topics. Each problem is worth the same number of points.

1. Let (X, A) be a pair of topological spaces, and consider the union $X \cup CA$ of X with the cone on A. Here we think of $CA = A \times [0,1]/\{(a,1),\, a \in A\}$, and in $X \cup CA$ we identify $(a,0) \in CA$ with $a \in A \subset X$.

   a) Prove that X is a retract of $X \cup CA$ if and only if A is contractible in X, i.e., there is a homotopy from the inclusion $A \hookrightarrow X$ to a constant map.
   b) Prove that $(X \cup CA)/X$ is homeomorphic to the suspension SA.

2. Let $A = \{(0,0,t) \mid -2 \leq t \leq 2\} \subseteq \mathbb{R}^3$ and $B = \{x \in \mathbb{R}^3 \mid |x| = 2\} \subseteq \mathbb{R}^3$.

   a) Compute $\pi_1(A \cup B)$.
   b) Compute the homology groups of $A \cup B$.

3. What are all the connected covering spaces of the 2-dimensional torus T^2, up to equivalence? Justify your answer.

4. Consider a commutative diagram of abelian groups

$$\begin{tikzcd}
B_1 & & B_2 \\
& C \arrow[ul, "j_2"'] \arrow[ur, "j_1"] & \\
A_1 \arrow[uu, "f_1"] \arrow[ur, "i_1"'] & & A_2 \arrow[uu, "f_2"'] \arrow[ul, "i_2"]
\end{tikzcd}$$

such that $j_1 i_1 = 0 = j_2 i_2$. Prove that if f_1 and f_2 are isomorphisms and the sequence

$$A_2 \xrightarrow{i_2} C \xrightarrow{j_2} B_1$$

is exact, then the maps

$$i_1 + i_2 \colon A_1 \oplus A_2 \to C \text{ and } (j_2, j_1) \colon C \to B_1 \oplus B_2$$

are isomorphisms and that the sequence

$$A_1 \xrightarrow{i_1} C \xrightarrow{j_1} B_2$$

is exact.

5. Let $M \subset \mathbb{R}^3$ be defined by the equations

$$x^2 + y^2 - z^2 = 1$$
$$z^2 - xy = 0.$$

Prove that M is a smooth manifold and find its dimension.

6. Let M be a smooth compact manifold of dimension n, and $f : M \to \mathbb{R}^{n+1} - \{0\}$ a smooth map. Prove that there exists a line through the origin in $\mathbb{R}^{n+1}$ that intersects $f(M)$ in only finitely many points.

7. Prove that if M is any smooth manifold, then the tangent bundle TM is an orientable manifold.

8. Prove that any continuous map $f : S^n \to S^n$ with $\deg(f) \neq (-1)^{n+1}$ has a fixed point. As a suggestion, you might consider proving the contrapositive.