Topology General Exam **August 21, 2019**

Student ID: ______________________

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style. All work should be your own—no outside sources are permitted—using methods and results from the first year topology course topics. Each problem is worth the same number of points.

1. Let $f\colon S^n \to S^n$ be a smooth map with the property that $df_x\colon T_xS^n \to T_{f(x)}S^n$ is injective for every $x \in S^n$.

a) Prove that f is a diffeomorphism provided that $n \geq 2$.

b) Find a counterexample to part (a) in the case that $n = 1$.

2. Let M be a connected smooth manifold of dimension m and let $N \subset M$ be a smooth submanifold of dimension $n = m - k$.

a) Show that if $k \geq 2$ then $M \setminus N$ is connected.

b) Prove that if $k \geq 3$ then the inclusion $M \setminus N \to M$ induces an isomorphism $\pi_1(M \setminus N, x_0) \to \pi_1(M, x_0)$ for $x_0 \in M \setminus N$ any basepoint.

3. Let M be a smooth compact manifold with nonempty boundary $N = \partial M$. Prove that there does not exist a retraction $M \to N$.

4. Let $M_n \cong \mathbb{R}^{n^2}$ denote the square $n \times n$ matrices with real entries.
   a) Prove that $\mathrm{SL}_n(\mathbb{R})$, the matrices with determinant 1, is a smooth submanifold of M_n.
   b) Prove that $\mathrm{SL}_n(\mathbb{R})$ is not compact.

5. Let K be a 2–complex consisting of one 0–cell, two 1–cells labelled $\{a, b\}$, and a single 2–cell attached along the loop $aba^{-1}b$.

a) Find $H_*(K; \mathbb{Z})$.
b) Find $H_*(K; \mathbb{Z}/2\mathbb{Z})$.
c) Show that K is homeomorphic to a smooth manifold.

6. Lef $f\colon S^n \to S^n$ be a continuous map.

a) Prove that if f has no fixed points then f is homotopic to the antipodal map.

b) Let G be a group acting freely on S^n, where here n is even. Prove that G has order at most two. [Hint: construct a homomorphism to $\mathbb{Z}/2\mathbb{Z}$ using degree and argue that this map is injective.]

7. Let X be the 1-point union of two copies of the projective plane, $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$.

   a) Find a presentation for $\pi_1(X)$ and describe $H_1(X;\mathbb{Z})$ as a direct sum of cyclic groups.

   b) Prove that every continuous map $f : X \to S^1$ is nullhomotopic.

8. Let C be a circle that is smoothly embedded in $\mathbb{R}^3$. For example C could be the trefoil knot:

but your methods should not be specific to this case. Find the homology groups $H_*(\mathbb{R}^3 \setminus C; \mathbb{Z})$.