Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style. All work should be your own – no outside sources are permitted – using methods and results from the first year topology course topics.

1. Let $f : M \to N$ be a smooth map between manifolds of dimension $m$ and $n$, respectively. Let $q \in N$ be a regular value for $f$, and let $X = f^{-1}(q) \subset M$.
   a) Prove that if $M$ is orientable, then $X$ is orientable.
   b) Let $g : K \to M$ be another smooth map, where $K$ is a smooth manifold. Prove that $q \in N$ is a regular value for $f \circ g$ if and only if $g$ is transverse to $X$. 
2. a) Prove that any smooth map \( f : S^k \to \mathbb{R}^n \) can be extended to a smooth map \( F : D^{k+1} \to \mathbb{R}^n \), where \( S^k = \partial D^{k+1} \) is the \( k \)-dimensional sphere and \( D^{k+1} \) the unit ball of dimension \( k + 1 \).

b) Let \( M \subset \mathbb{R}^n \) be a smooth compact manifold of dimension \( m \), and assume \( k < n - m - 1 \). Prove that any smooth map \( f : S^k \to \mathbb{R}^n - M \) can be extended to a smooth map \( F : D^{k+1} \to \mathbb{R}^n - M \).
3. Let $M$ and $N$ be the subsets of $\mathbb{R}^3$ defined by

$$M = \{x^2 + y^2 + z^2 = 1\} \quad N = \{x^2 - y^2 + z^2 = c\}$$

for a real number $c$. Justify your responses to the following:

a) Determine all values of $c$ for which $M$ and $N$ are submanifolds of $\mathbb{R}^3$, and the intersection $M \cap N$ is transverse.

b) Determine all values of $c$ for which $M \cap N$ is a submanifold of $\mathbb{R}^3$. 
4. Let $X$ and $Y$ be closed, compact, oriented manifolds of the same dimension. Let $f, g : X \to Y$ be two smooth maps. The graphs of $f$ and $g$ are the submanifolds of $X \times Y$ given by

$$
\Gamma_f = \{(x, f(x)) \mid x \in X\} \quad \Gamma_g = \{(x, g(x)) \mid x \in X\},
$$
oriented so that the diffeomorphisms $X \to \Gamma_f$ and $X \to \Gamma_g$ given by $x \mapsto (x, f(x))$ and $x \mapsto (x, g(x))$ are orientation-preserving.

The coincidence number of $f$ and $g$, written $C(f, g)$, is defined to be the intersection number $I(\Gamma_f, \Gamma_g) \in \mathbb{Z}$ (sometimes also written $\Gamma_f.A \Gamma_g$).

a) Prove that if $C(f, g) \neq 0$ then for any smooth maps $\tilde{f}, \tilde{g} : X \to Y$ such that $\tilde{f}$ and $\tilde{g}$ are homotopic to $f$ and $g$, respectively, there exists a point $x \in X$ such that $\tilde{f}(x) = \tilde{g}(x)$.

b) Let $f, g : S^1 \to S^1$ be two smooth maps of degree $n$ and $m$, respectively. Prove that if $n \neq m$, then there is a point $x \in S^1$ with $f(x) = g(x)$. 
5. Let $ p : S^n \to \mathbb{R}P^n $ be the projection.
   a) Let $ f : \mathbb{R}P^n \to \mathbb{R}P^n $ be continuous. Show that then there exists a continuous map $ \tilde{f} : S^n \to S^n $ such that $ p \circ \tilde{f} = f \circ p : S^n \to \mathbb{R}P^n $. 
   b) Show that every continuous map $ f : \mathbb{R}P^{2k} \to \mathbb{R}P^{2k} $ has a fixed point.
6. Suppose given a commutative diagram of abelian groups with exact rows:

\[
\begin{array}{ccccccc}
C_1 & \xrightarrow{\alpha_1} & C_2 & \xrightarrow{\alpha_2} & C_3 & \xrightarrow{\alpha_3} & C_4 & \xrightarrow{\alpha_4} & C_5 \\
\downarrow{f_1} & & \downarrow{f_2} & & \downarrow{f_3} & & \downarrow{f_4} & & \downarrow{f_5} \\
D_1 & \xrightarrow{\beta_1} & D_2 & \xrightarrow{\beta_2} & D_3 & \xrightarrow{\beta_3} & D_4 & \xrightarrow{\beta_4} & D_5.
\end{array}
\]

Prove part of the Five Lemma: show that if \( f_2 \) and \( f_4 \) are monomorphisms, and \( f_1 \) is an epimorphism, then \( f_3 \) is a monomorphism. (Hint: start by showing that \( x \in \text{Ker}(f_3) \Rightarrow x \in \text{Ker}(\alpha_3) \).)
7. Let $C \subset \mathbb{R}^3$ be the union of the $x$–axis and the $y$–axis. Compute $H_*(\mathbb{R}^3 - C; \mathbb{Z})$. (Hint: note that $\mathbb{R}^3 - C = (\mathbb{R}^3 - x$-axis) $\cap (\mathbb{R}^3 - y$-axis).)
8. Let $a : S^1 \to S^1 \vee S^1$ and $b : S^1 \to S^1 \vee S^1$ respectively be the inclusion of the circle as the first and second wedge summand. Then $\pi_1(S^1 \vee S^1)$ can be identified with the free group on $a$ and $b$. Let $f : S^1 \to S^1$ represent the element $c \in \pi_1(S^1 \vee S^1)$, and let $X_f = (S^1 \vee S^1) \cup_f D^2$, the topological space obtained by identifying points on $S^1 = \partial D^2$ with their images under $f$ in $S^1 \vee S^1$.

If $c = a(ab)^4a$, give a presentation of $\pi_1(X_f)$ and compute $H_*(X_f; \mathbb{Z})$, describing the homology groups as direct sums of cyclic groups, as usual.