Topology General Exam **January 9, 2019** **Student ID:** ____________________

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style. All work should be your own – no outside sources are permitted – using methods and results from the first year topology course topics.

1. Let $f : M \to N$ be a smooth map between manifolds of dimension m and n, respectively. Let $q \in N$ be a regular value for f, and let $X = f^{-1}(q) \subset M$.
   a) Prove that if M is orientable, then X is orientable.
   b) Let $g : K \to M$ be another smooth map, where K is a smooth manifold. Prove that $q \in N$ is a regular value for $f \circ g$ if and only if g is transverse to X.

2. a) Prove that any smooth map $f : S^k \to \mathbb{R}^n$ can be extended to a smooth map $F : D^{k+1} \to \mathbb{R}^n$, where $S^k = \partial D^{k+1}$ is the k-dimensional sphere and D^{k+1} the unit ball of dimension $k+1$.

   b) Let $M \subset \mathbb{R}^n$ be a smooth compact manifold of dimension m, and assume $k < n - m - 1$. Prove that any smooth map $f : S^k \to \mathbb{R}^n - M$ can be extended to a smooth map $F : D^{k+1} \to \mathbb{R}^n - M$.

3. Let M and N be the subsets of $\mathbb{R}^3$ defined by

$$M = \{x^2 + y^2 + z^2 = 1\} \qquad N = \{x^2 - y^2 + z^2 = c\}$$

for a real number c. Justify your responses to the following:

a) Determine all values of c for which M and N are submanifolds of $\mathbb{R}^3$, and the intersection $M \cap N$ is transverse.

b) Determine all values of c for which $M \cap N$ is a submanifold of $\mathbb{R}^3$.

4. Let X and Y be closed, compact, oriented manifolds of the same dimension. Let $f, g : X \to Y$ be two smooth maps. The graphs of f and g are the submanifolds of $X \times Y$ given by

$$\Gamma_f = \{(x, f(x)) \,|\, x \in X\} \qquad \Gamma_g = \{(x, g(x)) \,|\, x \in X\},$$

oriented so that the diffeomorphisms $X \to \Gamma_f$ and $X \to \Gamma_g$ given by $x \mapsto (x, f(x))$ and $x \mapsto (x, g(x))$ are orientation-preserving.

The *coincidence number* of f and g, written $C(f, g)$, is defined to be the intersection number $I(\Gamma_f, \Gamma_g) \in \mathbb{Z}$ (sometimes also written $\Gamma_f.\Gamma_g$).

a) Prove that if $C(f, g) \neq 0$ then for any smooth maps $\tilde{f}, \tilde{g} : X \to Y$ such that $\tilde{f}$ and $\tilde{g}$ are homotopic to f and g, respectively, there exists a point $x \in X$ such that $\tilde{f}(x) = \tilde{g}(x)$.

b) Let $f, g : S^1 \to S^1$ be two smooth maps of degree n and m, respectively. Prove that if $n \neq m$, then there is a point $x \in S^1$ with $f(x) = g(x)$.

5. Let $p : S^n \to \mathbb{R}P^n$ be the projection.
   a) Let $f : \mathbb{R}P^n \to \mathbb{R}P^n$ be continuous. Show that then there exists a continuous map $\tilde{f} : S^n \to S^n$ such that $p \circ \tilde{f} = f \circ p : S^n \to \mathbb{R}P^n$.
   b) Show that every continuous map $f : \mathbb{R}P^{2k} \to \mathbb{R}P^{2k}$ has a fixed point.

6. Suppose given a commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccccc}
C_1 & \xrightarrow{\alpha_1} & C_2 & \xrightarrow{\alpha_2} & C_3 & \xrightarrow{\alpha_3} & C_4 & \xrightarrow{\alpha_4} & C_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
D_1 & \xrightarrow{\beta_1} & D_2 & \xrightarrow{\beta_2} & D_3 & \xrightarrow{\beta_3} & D_4 & \xrightarrow{\beta_4} & D_5.
\end{array}$$

Prove part of the Five Lemma: show that if f_2 and f_4 are monomorphisms, and f_1 is an epimorphism, then f_3 is a monomorphism. (Hint: start by showing that $x \in Ker(f_3) \Rightarrow x \in Ker(\alpha_3)$.)

7. Let $C \subset \mathbb{R}^3$ be the union of the x–axis and the y–axis. Compute $H_*(\mathbb{R}^3 - C; \mathbb{Z})$. (Hint: note that $\mathbb{R}^3 - C = (\mathbb{R}^3 - x\text{-axis}) \cap (\mathbb{R}^3 - y\text{-axis})$.)

8. Let $a : S^1 \to S^1 \vee S^1$ and $b : S^1 \to S^1 \vee S^1$ respectively be the inclusion of the circle as the first and second wedge summand. Then $\pi_1(S^1 \vee S^1)$ can be identified with the free group on a and b. Let $f : S^1 \to S^1$ represent the element $c \in \pi_1(S^1 \vee S^1)$, and let $X_f = (S^1 \vee S^1) \cup_f D^2$, the topological space obtained by identifying points on $S^1 = \partial D^2$ with their images under f in $S^1 \vee S^1$.

If $c = a(ab)^4 a$, give a presentation of $\pi_1(X_f)$ and compute $H_*(X_f; \mathbb{Z})$, describing the homology groups as direct sums of cyclic groups, as usual.