Topology General Exam **August 22, 2018** **Student ID:** ______________________

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style. All work should be your own – no outside sources are permitted – using methods and results from the first year topology course topics.

1. Let M be a smooth n-manifold and $f : M \to M$ a smooth map. Let

$$\Gamma_f = \{(x, f(x)) \,|\, x \in M\} \quad \text{and} \quad \Delta = \{(x, x) \,|\, x \in M\}$$

be the graph of f and the diagonal submanifold, respectively, in $M \times M$.

Prove that if $x_0 \in M$ is a point with $f(x_0) = x_0$, then the following are equivalent:

(i) The manifolds Γ_f and Δ intersect transversely at (x_0, x_0).

(ii) The linear map $df_{x_0} : T_{x_0}M \to T_{x_0}M$ does not have 1 as an eigenvalue.

2. Let S be an oriented surface of genus 2 without boundary.

(a) By describing S as a polygon with certain pairs of edges identified and using the Seifert–Van Kampen theorem, or by another method, give a presentation for the group $\pi_1(S)$.

(b) Show that if a finite G group acts freely on S, then G must have order either 1 or 2.

3. Let X be a smooth, compact n-manifold with boundary ∂X. Prove that there does not exist a retraction $X \to \partial X$, that is, show there exists no smooth map $f : X \to \partial X$ with $f(x) = x$ for all $x \in \partial X$.

4. (a) Let M and N be smooth connected closed (= compact without boundary) manifolds of the same dimension. Show that a submersion $f : M \to N$ will then be a finite sheeted covering map.

(b) Explain why if M is a connected closed surface, and $f : M \to S^2$ is a submersion, then f must, in fact, be a diffeomorphism.

5. Let C_* and D_* be chain complexes of abelian groups.

(a) Complete the definition: Two chain maps $f_*, g_* : C_* \to D_*$ are *chain homotopic* if ....

(b) Show that if f_* is chain homotopic to g_*, and g_* is chain homotopic to h_*, then f_* is chain homotopic to h_*.

(c) Prove that if $f_*, g_* : C_* \to D_*$ are chain homotopic chain maps, then

$$H(f_*) = H(g_*) : H_*(C_*) \to H_*(D_*).$$

6. Suppose X is a CW complex with n–skeleton X_n for $n \geq 0$.

(a) Define the associated cellular chain complex: the groups $C_n^{CW}(X)$ and the differentials $d_n^{CW} : C_{n+1}^{CW}(X) \to C_n^{CW}(X)$. Then prove that $d_{n-1}^{CW} \circ d_n^{CW} = 0$.

(b) Explain why $C_n^{CW}(X)$ is isomorphic to a free abelian group with one generator for each n–cell of X.

7. Prove that the subset of $\mathbb{R}^3$ determined by the following equations is a manifold:

$$\begin{aligned} 2x^2 + 3y + z &= 6 \\ -x^2 + 2y^3 + z^2 &= 2 \end{aligned}$$

Also describe the tangent space to this manifold at the point $(1, 1, 1)$.

8. Let $S^3 \overset{p_1}{\longleftarrow} S^3 \vee S^3 \overset{p_2}{\longrightarrow} S^3$ be the two 'projection maps': the other sphere is collapsed to the basepoint. Then say that a map $f : S^3 \to S^3 \vee S^3$ has type (m, n) if the degree of $p_1 \circ f$ is m and the degree of $p_2 \circ f$ is n. Let $X_f = (S^3 \vee S^3) \cup_f D^4$.

Compute the homology groups of X_f if f has type $(8, 6)$, describing the homology groups as direct sums of cyclic groups, as usual.