1. Let $M$ be a smooth $n$-manifold and $f : M \to M$ a smooth map. Let

$$\Gamma_f = \{(x, f(x)) \mid x \in M\} \quad \text{and} \quad \Delta = \{(x, x) \mid x \in M\}$$

be the graph of $f$ and the diagonal submanifold, respectively, in $M \times M$.

Prove that if $x_0 \in M$ is a point with $f(x_0) = x_0$, then the following are equivalent:

(i) The manifolds $\Gamma_f$ and $\Delta$ intersect transversely at $(x_0, x_0)$.

(ii) The linear map $df_{x_0} : T_{x_0}M \to T_{x_0}M$ does not have 1 as an eigenvalue.
2. Let $S$ be an oriented surface of genus 2 without boundary.

(a) By describing $S$ as a polygon with certain pairs of edges identified and using the Seifert–Van Kampen theorem, or by another method, give a presentation for the group $\pi_1(S)$.

(b) Show that if a finite $G$ group acts freely on $S$, then $G$ must have order either 1 or 2.
3. Let $X$ be a smooth, compact $n$-manifold with boundary $\partial X$. Prove that there does not exist a retraction $X \to \partial X$, that is, show there exists no smooth map $f : X \to \partial X$ with $f(x) = x$ for all $x \in \partial X$. 


4. (a) Let $M$ and $N$ be smooth connected closed (= compact without boundary) manifolds of the same dimension. Show that a submersion $f : M \to N$ will then be a finite sheeted covering map.

(b) Explain why if $M$ is a connected closed surface, and $f : M \to S^2$ is a submersion, then $f$ must, in fact, be a diffeomorphism.
5. Let $C_*$ and $D_*$ be chain complexes of abelian groups.

(a) Complete the definition: Two chain maps $f_*, g_* : C_* \to D_*$ are chain homotopic if . . . .

(b) Show that if $f_*$ is chain homotopic to $g_*$, and $g_*$ is chain homotopic to $h_*$, then $f_*$ is chain homotopic to $h_*$. 

(c) Prove that if $f_*, g_* : C_* \to D_*$ are chain homotopic chain maps, then

\[ H(f_*) = H(g_*) : H_*(C_*) \to H_*(D_*) \]
6. Suppose $X$ is a CW complex with $n$–skeleton $X_n$ for $n \geq 0$.

   (a) Define the associated cellular chain complex: the groups $C_{n}^{CW}(X)$ and the differentials $d_{n}^{CW} : C_{n+1}^{CW}(X) \rightarrow C_{n}^{CW}(X)$. Then prove that $d_{n-1}^{CW} \circ d_{n}^{CW} = 0$.

   (b) Explain why $C_{n}^{CW}(X)$ is isomorphic to a free abelian group with one generator for each $n$–cell of $X$. 
7. Prove that the subset of \( \mathbb{R}^3 \) determined by the following equations is a manifold:

\[
\begin{align*}
2x^2 + 3y + z &= 6 \\
-x^2 + 2y^3 + z^2 &= 2
\end{align*}
\]

Also describe the tangent space to this manifold at the point \((1, 1, 1)\).
8. Let \( S^3 \xleftarrow{p_1} S^3 \vee S^3 \xrightarrow{p_2} S^3 \) be the two ‘projection maps’: the other sphere is collapsed to the basepoint. Then say that a map \( f : S^3 \to S^3 \vee S^3 \) has type \( (m, n) \) if the degree of \( p_1 \circ f \) is \( m \) and the degree of \( p_2 \circ f \) is \( n \). Let \( X_f = (S^3 \vee S^3) \cup_f D^4 \).

Compute the homology groups of \( X_f \) if \( f \) has type \((8, 6)\), describing the homology groups as direct sums of cyclic groups, as usual.