Topology General Exam
August 26, 2017

Name: ______________________________

Instructions: This is a four hour exam and 'closed book'. There are eight problems. Show your work using methods and results from the first year topology course topics. Results from one part of a problem can be assumed in later parts.

1. Let $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^3$ be given by $f(x,y) = (x^2, y^2, xy)$.

(a) Prove that f is an immersion, but not an embedding.

(b) Find all values of c such that f is transverse to the plane $\{(u,v,w) \mid v = c\} \subset \mathbb{R}^3$.

2. Two covering spaces over X, $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, are said to be isomorphic if there exists a homeomorphism $h : X_1 \to X_2$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\quad h \quad} & X_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

commutes.

Describe the isomorphism classes of covers of the space $\mathbb{R}P^2 \times \mathbb{R}P^2$.

3. Suppose that $t \in \mathbb{R}$ is a regular value of a smooth map $f : \mathbb{R}^n \to \mathbb{R}$, and let $M = f^{-1}(t)$.

(a) Will M necessarily have a nowhere vanishing normal vector field?

(b) Will M necessarily have a nowhere vanishing tangent vector field?

(c) Will M necessarily be orientable?

In each part, explain how to construct the vector field or orientation if it must exist, and give a counterexample, if it needn't.

4. **(a)** Define what it means for two chain maps between chain complexes to be *chain homotopic*. Then prove that if $f_*, g_* : C_* \to D_*$ are chain homotopic, then $f_* = g_* : H_*(C_*) \to H_*(D_*)$.

(b) Call a chain map a *quasi-isomorphism* if it induces an isomorphism on homology. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle g_*} & & \big\downarrow{\scriptstyle h_*} & & \\ 0 & \longrightarrow & D_* & \longrightarrow & E_* & \longrightarrow & F_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of chain complexes, such that each horizontal row is exact. Show that if f_* and g_* is a quasi-isomorphism then so is h_*.

5. Let $f : M \to N$ be a smooth map between manifolds of dimension m and n respectively. Let $D(f) \subset M \times M$ be the 'double point' subspace:

$$D(f) = \{(x, y) \mid x \neq y \text{ and } f(x) = f(y)\}.$$

(a) Say that f is *self transverse* if for all $(x, y) \in D(f)$, $d_x f(T_x M) + d_y f(T_y M) = T_{f(x)} N$. Show that then $D(f)$ is a smooth manifold of $M \times M$, and find its dimension. [Hint: Show that if f is self transverse, then the function $f \times f : M \times M - \Delta(M) \to N \times N$ is transverse to the diagonal $\Delta(N) \subset N \times N$.]

(b) Describe, with pictures and/or words, an example of a self transverse smooth function $f : S^1 \to \mathbb{R}^2$ for which $D(f)$ is nonempty.

(c) Describe, with pictures and/or words, an example of a smooth function $f : S^1 \to \mathbb{R}^2$ that is not self transverse.

6. (a) Show that there is no continuous map $g : S^2 \to S^2$ such that, for all $x \in S^2$, $g(x) \neq x$ and $g(x) \neq -x$. [Hint: if such a g exists, explain how it can be used to show that the antipodal map is homotopic to the identity map on S^2. Hmm ...]

(b) Show that every continuous map $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ has a fixed point. [If f has no fixed point, use covering space theory to show that one can construct g as in part (a).]

7. Suppose a finite group G acts freely on the right of a Hausdorff space X. Let X/G be the space of G–orbits – sets of the form $\{xg \mid g \in G\}$ – given the quotient topology.

(a) Show that the quotient map $X \to X/G$ is a covering space map. [Hint: you need to show that each point $x \in X$ has an open neighborhood U such that all the translates of U – the sets Ug for $g \in G$ – are disjoint.]

(b) How are the groups G, $\pi_1(X)$ and $\pi_1(X/G)$ related?

(c) Show that there exists a smooth 3-manifold with a fundamental group that is both finite and non-abelian. [One approach: Recall that S^3 can be viewed as the quaternions of unit length.]

8. (a) S^2 has a CW complex structure with one 0-cell and one 2-cell, and then the associated 'product' CW structure on $S^2 \times S^2$ has one 0-cell, two 2-cells, and one 4-cell. Compute the homology groups of $S^2 \times S^2$.

(b) If M is a smooth connected n–dimensional manifold, let $\widetilde{M}$ denote M with a small open n–ball removed. Show that $\widetilde{S^2 \times S^2}$ is homotopy equivalent to $S^2 \vee S^2$. [Hint: you can assume that the small open 4–ball is removed from the interior of the 4-cell.]

(c) The connected sum $M\#N$ of two n–manifolds admits a decomposition $M\#N = \widetilde{M} \cup \widetilde{N}$ with $\bar{M} \cap \bar{N}$ diffeomorphic to S^{n-1}. Compute the homology groups of $(S^2 \times S^2)\#(S^2 \times S^2)$.