Topology General Examination **January 13, 2017**

Solve the following problems on your own paper. Be sure your solutions are legible and clearly organized, written in complete sentences in good mathematical style. All work should be your own; no outside sources are permitted. You may use without proof standard results from first-semester differential and algebraic topology, unless otherwise indicated; where appropriate you should cite theorems by name.

1. Consider three distinct points p_1, p_2, p_3 in the disk D^2. Let X be the topological space X obtained from D^2 by identifying these three points, $X = D^2/p_1 \sim p_2 \sim p_3$. Compute the fundamental group of X, and construct two different connected double covers of X.

2. Let $p(z)$ be a polynomial with complex coeffients having even degree. Prove that for all sufficiently large r, there exists a continuous function $f : \{|z| = r\} \to \mathbb{C}$ that is a square root of p on the circle $|z| = r$. That is, f satisfies $f(z)^2 = p(z)$ for all z with $|z| = r$.

3. Let $f\colon S^1 \longrightarrow S^1 \times [0,1]$ be a map such that the composition of f with the projection $p\colon S^1 \times [0,1] \longrightarrow S^1$ has $|\deg(f)| > 1$. Prove that f is not an embedding.

4. Consider $\mathbb{R}P^2$ as the set of lines through the origin in $\mathbb{R}^3$ with the usual topology. Let A be a 1-dimensional submanifold of the unit sphere $S^2 \subset \mathbb{R}^3$, and define the subset $B \subset \mathbb{R}P^2$ consisting of the lines intersecting A. Is B necessarily a submanifold of $\mathbb{R}P^2$? If your answer is 'Yes', give a proof; otherwise give a counterexample.

5. Let C_1, C_2 be two simple closed curves, disjointly embedded as smooth submanifolds in $\mathbb{R}^2$. Prove that there exist points $p \in C_1, q \in C_2$ such that the line through p, q is transverse to both C_1 and C_2.

6. Consider the special linear group $SL_2(\mathbb{R})$ of real 2×2 matrices of determinant 1. Find the tangent space of $SL_2(\mathbb{R})$ at the identity matrix I (give a description of the tangent space as a set of matrices). Find the differential $d_I f$ at I of the map $f : SL_2(\mathbb{R}) \longrightarrow SL_2(\mathbb{R})$, given by $f(A) = A^2$.

7. Recall that for relatively prime integers $p > q \geq 1$ the 3-dimensional lens space $L(p,q)$ has homology groups given by

$$H_i(L(p,q);\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 3 \\ 0 & i = 2 \\ \mathbb{Z}/p\mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

(note p and q need not be prime).

Fix a prime r, and use the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/r \to 0$ of coefficients to compute the homology $H_i(L(p,q);\mathbb{Z}/r)$ for all i.

(Exam continues)

8. Fix an integer $n \geq 2$ and let X be the space given by the quotient of the unit ball $B \subset \mathbb{R}^3$, where points on ∂B differing by a rotation by $2\pi/n$ around the z axis are identified. Describe a CW structure on X and the associated cellular chain complex, and use it to find the homology groups of X.

9. a) Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian, and find a basis.

b) Use part (a) and the Mayer-Vietoris sequence to compute the homology groups of the subspace of $[0,1] \times [0,1]$ consisting of the four boundary edges plus all points in the interior whose first coordinate is rational.