Topology General Examination **August 16, 2016**

Solve the following problems on your own paper. Be sure your solutions are legible and clearly organized, written in complete sentences in good mathematical style. All work should be your own; no outside sources are permitted. You may use without proof standard results from first-semester differential and algebraic topology, unless otherwise indicated; where appropriate you should cite theorems by name.

1. Let $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$ be the complex exponential map; in particular recall that exp is a universal covering map. Let X be a topological space and $f : X \to \mathbb{C} - \{0\}$ a continuous function. A *logarithm* of f is a continuous function $g : X \to \mathbb{C}$ such that $\exp \circ g = f$. If X is path connected and locally path connected, and if the fundamental group of X is finite, prove that any continuous function $f : X \to \mathbb{C} - \{0\}$ has a logarithm.

2. Let $f : S^2 \longrightarrow \mathbb{R}^2$ be a smooth map. Show that the cardinality of $f^{-1}(y)$ is even for any regular value y.

3. Let S be the surface in $\mathbb{R}^3$, defined by the equation $z = x^2 + y^2$. Consider the set C of points (x, y, z) on S such that the line joining the points $(1, 0, 0)$ and (x, y, z) is tangent to S. Describe C geometrically. Is C a submanifold of S? Justify your answer.

4. Let P be a regular hexagon in the plane with vertices A, B, C, D, E, F (labeled counterclockwise), together with its interior. Let X be the topological space obtained by identifying the two oriented edges AB and BC of P, and also identifying the four oriented edges CD, DE, EF, FA. Describe a CW structure on X, describe the cellular chain complex, and use it to calculate the homology groups of X. Express the homology groups in each dimension as the direct sum of a free abelian group and a finite abelian group.

5. Given a commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 & & & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
\end{array}$$

Prove that there exists a unique homomorphism $u : A \to A'$ making the diagram commute. (Give a full proof, do not quote general results.)

(exam continues)

6. Let $A = S^1 \times S^1$ be a torus, and $B = S^1 \times S^1 - D$ be a torus with a small open disk D removed. Let Y be the space obtained by attaching B to A using a map of degree $n \in \mathbb{Z}$ from the boundary circle of B to the circle $S^1 \times pt \subset A$.

a) Find a presentation of $\pi_1(Y)$ using the Seifert-Van Kampen theorem. Be sure to describe your generators and basepoint.

b) Find the homology groups of Y using the Mayer-Vietoris sequence.

7. Let X be the CW complex obtained from S^2 by attaching D^3 along a degree 2 map $\partial D^3 \longrightarrow S^2$. Similarly, define Y by attaching D^3 to S^2 along a degree 4 map.

Does the identity map $S^2 \longrightarrow S^2$ extend to a map $X \longrightarrow Y$? Does it extend to a map $Y \longrightarrow X$? Justify your answers.

8. Let M^m and N^n be closed manifolds. The graph of a smooth map $f : M \to N$ is the submanifold $G_f = \{(x, y) | y = f(x)\}$ of $M \times N$. Let $x \in M$, $y \in N$. For the following statements, give either a proof or a counterexample:

a) G_f is transverse to $M \times \{y\}$ for any f.

b) G_f is transverse to $\{x\} \times N$ for any f.