Topology General Exam
August 14, 2015

Name: ______________________________

Instructions: This is a four hour exam and 'closed book'. There are seven problems: the first 6 are worth 15 points and no. 7 is worth 10 points, for a maximum total of 100.

1. (a) Describe a connected double cover of $\mathbb{R}P^2 \vee S^1$. (There is more than one correct answer.)

(b) What are the homology groups of your double cover?

(c) What is the fundamental group of your double cover?

2. Suppose that $M \xrightarrow{g} N$ is a smooth maps between smooth manifolds of dimensions m and n respectively. Let $z \in N$ be a regular value for g and let $K = g^{-1}(z)$.

(a) Explain why K will be orientable if M is orientable.

(b) Now suppose that one also has a smooth map $L \xrightarrow{f} M$. Show that $z \in N$ will be a regular value for the composite $g \circ f$ if and only if f is transverse to K.

3. (a) Complete the definition: Two chain maps $f_*, g_* : C_* \to D_*$ are *chain homotopic* if ....

(b) Prove that if $f_*, g_* : C_* \to D_*$ are chain homotopic chain maps, then

$$H(f_*) = H(g_*) : H_*(C_*) \to H_*(D_*).$$

(c) Suppose that $h_* : D_* \to E_*$ is yet another chain map. Show that if f_* is chain homotopic to g_*, then the composite $h_* \circ f_*$ is chain homotopic to $h_* \circ g_*$.

4. (a) Show that an n–dimensional Lie group G is parallelizable, i.e., admits n smooth vector fields that are linearly independent when evaluated at any point.

(b) Prove that S^2 does not admit a group structure making it into a Lie group.

5. (a) Describe a smooth atlas for $\mathbb{R}P^3$.

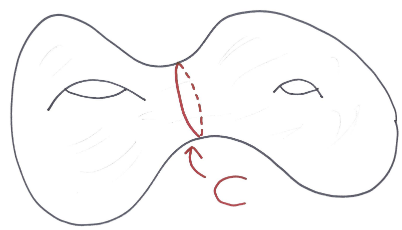
(b) Describe a C.W. complex structure for $\mathbb{R}P^3$.

(c) Describe the cellular chain complex associated to your answer to (b), and use this to compute $H_*(\mathbb{R}P^3)$.

6. (a) Let M and N be smooth connected closed (= compact without boundary) manifolds of the same dimension. Show that a submersion $f : M \to N$ will then be a finite sheeted covering map.

(b) Explain why if M is a connected closed surface, and $f : M \to S^2$ is a submersion, then f must, in fact, be a diffeomorphism.

7. Let C be the 'middle circle' in the genus 2 surface M as pictured:



Show that if $C' \subset M$ is any other embedded circle transverse to C, then C' intersects C in an even number of points.