Topology General Examination **January 9, 2015**

Solve the following problems on your own paper. Be sure your solutions are legible and clearly organized. All work should be your own; no outside sources are permitted. You may use without proof standard results from first-semester differential and algebraic topology. Be sure to state precisely the results that you are using.

In multiple part problems, late parts may depend on earlier ones. When working on later parts of such a problem, you may assume the results implied by earlier parts, even if you did not know how to do them.

1. Consider a smooth compact m-dimensional manifold embedded in Euclidean space, $M^m \subset \mathbb{R}^n$, $n > m$. The normal bundle of M is defined to be

$$N(M) = \{(m, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid m \in M \text{ and } v \in T_m M^{\perp}\}$$

Here $T_m M^{\perp} = \{v \in \mathbb{R}^n \mid \langle v, w\rangle = 0\ \forall w \in T_m M\}$, where $\langle \cdot, \cdot \rangle$ indicates the usual inner product on $\mathbb{R}^n$. Then $N(M)$ is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ of dimension n. There is a natural embedding $\iota : M \to N(M)$ given by $\iota(m) = (m, 0)$. Define $F : N(M) \to \mathbb{R}^n$ by $F(m, v) = m + v$.

(a) Given a point $m \in M$, we can choose a basis of $\mathbb{R}^n$ such that $T_m M = \mathbb{R}^m \times 0$ while $T_m M^{\perp} = 0 \times \mathbb{R}^{n-m}$. Prove that the differential DF is an isomorphism at the point $\iota(m) = (m, 0) \in N(M)$.

(b) Prove that there exist $\epsilon > 0$ such that the restriction of F to the subset $N_\epsilon(M) = \{(m, v) \in N(M) \mid |v| < \epsilon\}$ is a diffeomorphism onto a neighborhood of $M \subset \mathbb{R}^n$.

(c) Let M be the unit circle in $\mathbb{R}^2$. In this situation, what is the maximum possible value of ϵ from the previous part? Justify your answer.

2. Let D and D' denote two copies of the unit disk in $\mathbb{R}^2$, and introduce polar coordinates (r, θ) and (r', θ') on the two disks. For an integer $n \in \mathbb{Z}$, define an equivalence relation on the disjoint union $D \times S^1 \sqcup D' \times S^1$, generated by the relation

$$((r, \theta), \phi) \sim ((1 - r, \theta), \phi + n\theta) \qquad \text{for } 0 < |r| < 1.$$

Let Y_n denote the quotient $D \times S^1 \sqcup D' \times S^1 / \sim$.

(a) Prove that Y_n is a smooth compact 3-dimensional manifold.

(b) Use Van Kampen's theorem to find the fundamental group of Y_n.

3. Let G be a finite group acting on a smooth, compact, closed, oriented manifold M by orientation-preserving diffeomorphisms. Assume that the quotient space $N = M/G$ is a smooth manifold, and also assume that there exists a point $x \in M$ with the property that the group

$$G_x = \{g \in G \mid g \cdot x = x\}$$

is trivial. Prove that the natural quotient map $q : M \to N$ has degree equal to the order of G.

4. Let X and Y be closed, compact, oriented manifolds of the same dimension. Let $f, g : X \to Y$ be two smooth maps. The graphs of f and g are the submanifolds of $X \times Y$ given by

$$\Gamma_f = \{(x, f(x)) \,|\, x \in X\} \qquad \Gamma_g = \{(x, g(x)) \,|\, x \in X\},$$

oriented so that the obvious diffeomorphisms $X \to \Gamma_f$ and $X \to \Gamma_g$ given by $x \mapsto (x, f(x))$ and $x \mapsto (x, g(x))$ are orientation-preserving.

The *coincidence number* of f and g, written $C(f, g)$, is defined to be the intersection number $\Gamma_f . \Gamma_g \in \mathbb{Z}$.

(a) Prove that if $C(f, g) \neq 0$ then for any smooth maps $f', g' : X \to Y$ such that f' and g' are homotopic to f and g, respectively, there exists a point $x \in X$ such that $f'(x) = g'(x)$.

(b) Let $f, g : S^1 \to S^1$ be two maps of degree n and m, respectively. Prove that if $n \neq m$, then there is a point $x \in S^1$ with $f(x) = g(x)$.

5. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$. In this question, covers are assumed to be path connected.

(a) Prove that X does not have a normal 3-fold cover.

(b) Prove that X does have a non-normal 3-fold cover.

(c) Describe explicitly a 3-fold cover of X.

6. Let X be a connected CW-complex. For each $n \geq 0$, let X^n denote the n-skeleton of X. Prove that the inclusion $X^n \hookrightarrow X$ induces an isomorphism on π_1 for $n \geq 2$ and an epimorphism for $n = 1$.

7. Fix integers $0 \leq k < n$. Let $S^k \hookrightarrow S^n$ be the standard inclusion. Compute the homology of the space obtained from S^n by identifying antipodal points in S^k.

8. Let X be a compact Hausdorff space. Let us say that X is *cell-like* if it has the following property:

> For any embedding $f \colon X \hookrightarrow S^n$, the space $S^n \setminus f(X)$ has the same homology as the one-point space.

(a) (bonus question) Prove that if X is cell-like, then so is $X \times [0, 1]$. Note: At some point you may want to use a certain continuity property of singular homology. You can state it without proof.

Remember that regardless of what you did in part (a), you may assume it when working on subsequent parts.

(b) Prove that for each $k \geq 0$, the closed ball D^k is cell-like.

(c) Prove that for every embedding $f\colon S^k \hookrightarrow S^n$, the space $S^n \setminus f(S^k)$ has the same homology as S^{n-k-1}.

(d) Is $S^n \setminus f(S^k)$ necessarily homotopy equivalent to S^{n-k-1}? Either prove it or give a counterexample.