Topology General Examination **August 22, 2014**

Solve the following problems on your own paper. Be sure your solutions are legible and clearly organized. All work should be your own; no outside sources are permitted. You may use without proof standard results from first-semester differential and algebraic topology. Be sure to state precisely the results that you are using.

In multiple part problems, late parts may depend on earlier ones. When working on later parts of such a problem, you may assume the results implied by earlier parts, even if you did not know how to do them.

1. Define a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (y, x^3 - yx)$.

(a) What are the regular values of f?

(b) Find $\deg_{(1,0)}(f)$. (Note that here $(1, 0)$ is considered as a point of the range of f. Take $\mathbb{R}^2$ to be equipped with the standard orientation.)

2. Define a smooth map $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\varphi(x, y, z) = ((2 - \sqrt{x^2 + y^2})^2 + z^2,\ x^2 + y^2 - 4).$$

(a) Prove that the preimage $\varphi^{-1}(1, 0)$ is an embedded submanifold of $\mathbb{R}^3$, and give a verbal description of the submanifold.

(b) For what value(s) of c is the plane $x = c$ transverse to $\varphi^{-1}(1, 0)$? Justify your answer.

3. Suppose M, N are oriented, disjoint embedded submanifolds of $\mathbb{R}^3$ each diffeomorphic to a circle. The *linking number* between M and N is the integer $\ell k(M, N) \in \mathbb{Z}$ given by the degree of the map

$$\begin{array}{rcl} M \times N & \to & S^2 \\ (x, y) & \mapsto & \dfrac{x - y}{|x - y|}. \end{array}$$

Prove that if M is the boundary of a compact oriented surface S embedded in $\mathbb{R}^3 - N$, then $\ell k(M, N) = 0$.

4. Suppose $f : X^n \to Y^n$ is a smooth map between compact connected manifolds without boundary, and assume $df_x : T_xX \to T_{f(x)}Y$ is surjective for each $x \in X$.

(a) Prove that f is a covering map.

(b) Assume both X and Y are oriented. Prove that the degree of f as a covering map (number of sheets) is equal to $|\deg(f)|$, where $\deg(f)$ is the degree of f as a smooth map.

5. Prove that a continuous map $g : S^n \to S^n$ satisfying $g(x) = g(-x)$ for all $x \in S^n$ has even degree.

6. Let $p\colon \tilde{G} \to G$ be a homomorphism of topological groups that is also a covering map. Assume that $\tilde{G}$ is simply connected. Prove that $\pi_1(G, e)$ is isomorphic, as a group, to the kernel of p.

7. (a) Show that there is a free action of $\mathbb{Z}/4$ on the sphere S^3.

Even if you you did not do part (a), continue with the rest of the problem as if you constructed such an action.

Let $\mathbb{Z}/2$ act on S^2 by sending a point to its antipode. Taking cartesian product with the action of part (a), we get an action of $\mathbb{Z}/2 \times \mathbb{Z}/4$ on $S^2 \times S^3$. Let X be the quotient space of this action and let $q\colon S^2 \times S^3 \to X$ be the quotient map.

(b) Prove that q is a covering map.

(c) Describe the universal cover of X.

(d) What is the fundamental group of X?

(e) Describe all the connected covers of X, up to isomorphism. For each cover, specify its degree, its fundamental group, and its group of deck transformations.

8. Prove that if $n \geq 2$ then every continuous map $f\colon \mathbb{R}P^n \to S^1 \times S^1$ is homotopic to a constant.

9. Give an example (with justification) of a connected cover that is not regular.

10. (a) Suppose that we have inclusions of spaces $A \subset B \subset C$. Prove that there is a long exact sequence of relative homology groups

$$\cdots \to H_n(B, A) \to H_n(C, A) \to H_n(C, B) \to H_{n-1}(B, A) \to \cdots.$$

(b) Let D^n be the closed unit ball in $\mathbb{R}^n$. Let S^{n-1} be the unit sphere, and let D^{n-1}_+ be the "northern hemisphere" in S^{n-1}. We have inclusions $D^{n-1}_+ \subset S^{n-1} \subset D^n$. Prove that for every space X and every k there is a natural isomorphism

$$H_k(D^n \times X, S^{n-1} \times X) \cong H_{k-1}(S^{n-1} \times X, D^{n-1}_+ \times X).$$

(c) Prove that for every n and k there is an isomorphism

$$H_k(S^{n-1} \times X, D^{n-1}_+ \times X) \cong H_k(D^{n-1}_- \times X, S^{n-2} \times X)$$

Hint: excision.

(d) Conclude that for every n and X there is an isomorphism

$$H_k(D^n \times X, S^{n-1} \times X) \cong H_{k-n}(X).$$

(e) Prove that $H_k(D^n \times X, S^{n-1} \times X) \cong H_k(S^n \times X, * \times X)$

(f) Conclude that there is a natural isomorphism $H_k(S^n \times X) \cong H_k(X) \oplus H_{k-n}(X)$ Hint: The projection map $S^n \times X \to X$ is a retraction of the inclusion map $X \to S^n \times X$. What does it mean about the long exact sequence of the pair $(S^n \times X, * \times X)$?