Topology General Examination **January 10, 2014**

Solve the following problems on your own paper. Be sure your solutions are legible and clearly organized. All work should be your own; no outside sources are permitted. You may use without proof standard results from first-semester differential and algebraic topology; where appropriate you should cite theorems by name.

1. Suppose X is a connected, locally path-connected space whose fundamental group is finite. Prove that every continuous map from X to the torus T is null-homotopic.

2. Let T_1 and T_2 be two copies of the torus $S^1 \times S^1$, and let $f, g : S^1 \to S^1$ be two maps of degrees 2 and 6, respectively. Let $x_0 \in S^1$ be a fixed base point. Find the fundamental group of the space

$$X = T_1 \cup_F S^1 \times [0,1] \cup_G T_2,$$

where $F : S^1 \times \{0\} \to T_1$ is given by $F(x,0) = (f(x), x_0)$ and $G : S^1 \times \{1\} \to T_2$ is given by $G(x,1) = (g(x), x_0)$.

3. You are given the following commutative diagram of abelian groups:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1 & \stackrel{i}{\longrightarrow} & C_2 & \stackrel{j}{\longrightarrow} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Assume that all columns are exact, and that the first two rows are exact. Prove that the third row is exact at C_2, i.e., that $\text{im}(i) = \ker(j)$. (*Note:* in this situation the entire third row is exact, but you need not prove this.)

4. Consider a space X that is the union of two open subsets U and V such that:

- V is contractible
- both U and $U \cap V$ have the homology of a circle.

a) What are all the possibilities for the homology $H_*(X;\mathbb{Z})$?

b) Describe explicit spaces realizing all these possibilities.

5. Let M be a smooth compact manifold without boundary. Show that there is no submersion (i.e., smooth map whose differential is everywhere surjective) $F : M \longrightarrow \mathbb{R}^k$ for any $k > 0$.

6. Show that for any $n \geq 0$ the manifold $M = S^n \times \mathbb{R}$ is parallelizable (that is, its tangent bundle is trivial).

7. Let M be a smooth, closed (compact without boundary) n-dimensional submanifold of $\mathbb{R}^{n+1}$, with $0 \notin M$. Prove that there exists a line through 0 in $\mathbb{R}^{n+1}$ which intersects M in finitely many points (or is disjoint from M).

8. Let M and N be the subsets of $\mathbb{R}^3$ defined by

$$M = \{x^2 + y^2 + z^2 = 1\} \qquad N = \{x^2 - y^2 + z^2 = c\}$$

for a real number c. Justify your responses to the following:

a) Determine all values of c for which M and N are submanifolds of $\mathbb{R}^3$, and the intersection $M \cap N$ is transverse.

b) Determine all values of c for which $M \cap N$ is a submanifold of $\mathbb{R}^3$.

9. Suppose $f : S^n \longrightarrow S^n$, $n \geq 2$, is a smooth map whose differential is injective at each point. Prove that f is a diffeomorphism.