

Solve the following problems on your own paper. Be sure your solutions are legible and clearly organized. All work should be your own; no outside sources are permitted. You may use without proof standard results from first-semester differential and algebraic topology; where appropriate you should cite theorems by name.

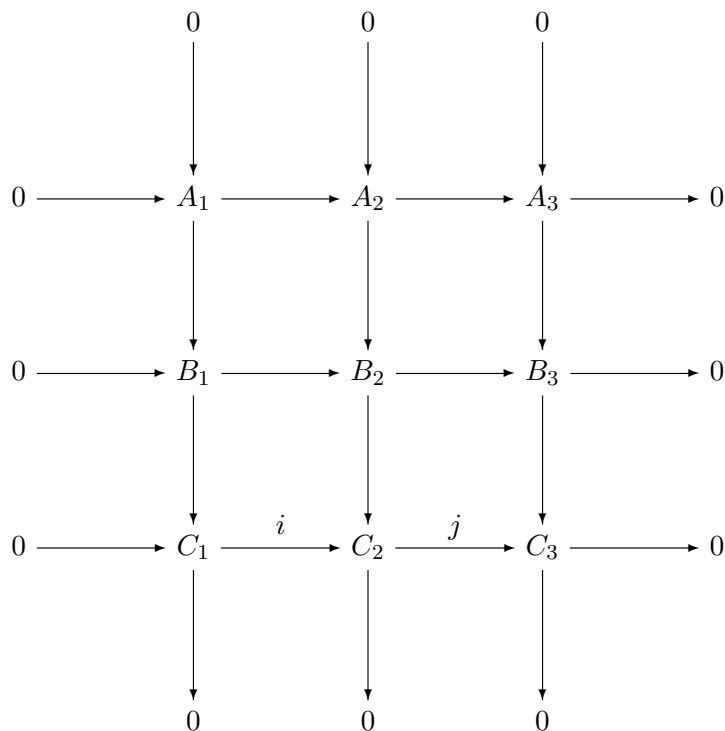
1. Suppose  $X$  is a connected, locally path-connected space whose fundamental group is finite. Prove that every continuous map from  $X$  to the torus  $T$  is null-homotopic.

2. Let  $T_1$  and  $T_2$  be two copies of the torus  $S^1 \times S^1$ , and let  $f, g : S^1 \rightarrow S^1$  be two maps of degrees 2 and 6, respectively. Let  $x_0 \in S^1$  be a fixed base point. Find the fundamental group of the space

$$X = T_1 \cup_F S^1 \times [0, 1] \cup_G T_2,$$

where  $F : S^1 \times \{0\} \rightarrow T_1$  is given by  $F(x, 0) = (f(x), x_0)$  and  $G : S^1 \times \{1\} \rightarrow T_2$  is given by  $G(x, 1) = (g(x), x_0)$ .

3. You are given the following commutative diagram of abelian groups:



Assume that all columns are exact, and that the first two rows are exact. Prove that the third row is exact at  $C_2$ , i.e., that  $\text{im}(i) = \text{ker}(j)$ . (Note: in this situation the entire third row is exact, but you need not prove this.)

4. Consider a space  $X$  that is the union of two open subsets  $U$  and  $V$  such that:

- $V$  is contractible
  - both  $U$  and  $U \cap V$  have the homology of a circle.
- a) What are all the possibilities for the homology  $H_*(X; \mathbb{Z})$ ?
- b) Describe explicit spaces realizing all these possibilities.
5. Let  $M$  be a smooth compact manifold without boundary. Show that there is no submersion (i.e., smooth map whose differential is everywhere surjective)  $F : M \rightarrow \mathbb{R}^k$  for any  $k > 0$ .
6. Show that for any  $n \geq 0$  the manifold  $M = S^n \times \mathbb{R}$  is parallelizable (that is, its tangent bundle is trivial).
7. Let  $M$  be a smooth, closed (compact without boundary)  $n$ -dimensional submanifold of  $\mathbb{R}^{n+1}$ , with  $0 \notin M$ . Prove that there exists a line through  $0$  in  $\mathbb{R}^{n+1}$  which intersects  $M$  in finitely many points (or is disjoint from  $M$ ).
8. Let  $M$  and  $N$  be the subsets of  $\mathbb{R}^3$  defined by
- $$M = \{x^2 + y^2 + z^2 = 1\} \quad N = \{x^2 - y^2 + z^2 = c\}$$
- for a real number  $c$ . Justify your responses to the following:
- a) Determine all values of  $c$  for which  $M$  and  $N$  are submanifolds of  $\mathbb{R}^3$ , and the intersection  $M \cap N$  is transverse.
- b) Determine all values of  $c$  for which  $M \cap N$  is a submanifold of  $\mathbb{R}^3$ .
9. Suppose  $f : S^n \rightarrow S^n$ ,  $n \geq 2$ , is a smooth map whose differential is injective at each point. Prove that  $f$  is a diffeomorphism.