Topology General Exam
August 24, 2012

Name: ______________________________

Instructions: This is a four hour exam and 'closed book'. There are eight problems.

1. (a) Suppose that $t \in \mathbb{R}$ is a regular value of a smooth map $f : \mathbb{R}^n \to \mathbb{R}$, and let $M = f^{-1}(t)$. Explain why M has a nowhere vanishing normal vector field.

(b) If $f(x, y, z) = x^2 + y^2 + z^2$, check that the hypothesis of part (a) holds when $t = 1$, and then draw a picture illustrating the conclusion.

2. Rigorously prove that the Möbius band is non–orientable.

3. (a) Let M and N be smooth connected closed (= compact without boundary) manifolds of the same dimension. Show that a submersion $f : M \to N$ will then be a finite sheeted covering map. (a submersion = a map whose differential is surjective at each point.)

(b) Explain why if M is a connected closed surface, and $f : M \to S^2$ is a submersion, then f must, in fact, be a diffeomorphism.

(c) Explain why if M is a connected closed surface, and $f : M \to S^1 \times S^1$ is a submersion, then M must be $S^1 \times S^1$.

4. Let $S^2 \xleftarrow{p_1} S^2 \vee S^2 \xrightarrow{p_2} S^2$ be the two 'projection maps': the other sphere is collapsed to the basepoint. Then say that a map $f : S^2 \to S^2 \vee S^2$ has type (m, n) if the degree of $p_1 \circ f$ is m and the degree of $p_2 \circ f$ is n. Let $X_f = (S^2 \vee S^2) \cup_f D^3$.

(a) Compute the homology groups of X_f if f has type $(4, 6)$, describing the homology groups as direct sums of cyclic groups, as usual.

(b) More generally, describe the homology groups of X_f if f has type (m, n).

5. Suppose that X is the union of open sets X_1 and X_2, and Y is the union of open sets Y_1 and Y_2. Let $f : X \to Y$ be a map that restricts to maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, and thus also $f_{12} : X_1 \cap X_2 \to Y_1 \cap Y_2$.

Prove that, if f_1, f_2 and f_{12} all induce isomorphisms in homology, then $f_* : H_*(X) \to H_*(Y)$ will also be an isomorphism.

6. Suppose $p : \tilde{Y} \to Y$ is a double cover. If X is a space such that $H_1(X)$ is a finite group of odd order, show that any map $f : X \to Y$ lifts through p: there exists $\tilde{f} : X \to \tilde{Y}$ such that $f = p \circ \tilde{f}$. (You can assume that X is locally 'friendly'.)

7. Let $M_2(\mathbb{R})$ be the vector space of all 2×2 real matrices, and let $f : M_2(\mathbb{R}) \to \mathbb{R}$ be given by $f(A) = \det(A)$. The differential of f at $A \in M_2(\mathbb{R})$ is a linear map $d_A f : M_2(\mathbb{R}) \to \mathbb{R}$.

(a) Compute $d_A f(A)$.

(b) Show that $SL_2(\mathbb{R})$, the group of 2×2 real matrices with determinant 1, is a smooth submanifold of $M_2(\mathbb{R})$.

(c) Show that $T_I SL_2(\mathbb{R})$, the tangent space of $SL_2(\mathbb{R})$ at the identity matrix I, is the subspace of $M_2(\mathbb{R})$ consisting of matrices with trace equal to 0.

8. Recall that the Brower Fixed Point Theorem says that every continuous self map of the closed n–ball D^n has a fixed point.

(a) Prove the theorem using homology.

(b) Prove the theorem using the methods of differential topology methods. (Step 1: If a continuous f had no fixed points, a nearby smooth function would also have no fixed points.)

Other ideas for problems:

Extra 1. (a) Describe a smooth atlas for $\mathbb{R}P^n$.

(b) Describe a C.W. complex structure for $\mathbb{R}P^n$.

Other problems suggested by Slava ...

Extra 2. View $\mathbb{R}P^n$ as the space of lines through the origin in $\mathbb{R}^{n+1}$. Show that, given a continuous map $f : \mathbb{R}P^n \to \mathbb{R}^{n+1} - \{0\}$, there exists $L \in \mathbb{R}P^n$ such that the vector $f(L)$ is orthogonal to the line L. (*hmm ... we need* $n > 0$.)

Nick's comments ... Alternative (and equivalent) Show that, for $n > 0$, there is no continuous map $f : \mathbb{R}P^n \to \mathbb{R}^{n+1} - \{0\}$ such that $f(L) \in L$ for all lines L.

Remark This seems to have a simple proof that doesn't involve any diff or alg topology: From such an f that shouldn't exist, one gets $g : S^n \to S^n$ such that (i) $g(x)$ is either x or $-x$ for all x, and (ii) $g(x) = g(-x)$ for all x. Since S^n is connected, (i) implies that g is either the identity or the antipodal map, and neither of these satisfy (ii).

Extra 3. Consider a smooth map $f\colon S^3 \longrightarrow S^2$ and let $x, y \in S^2$ be two regular values.

(a) Explain how orientations on the spheres S^2, S^3 induce an orientation of the 1-dimensional submanifolds $f^{-1}(x), f^{-1}(y) \subset S^3$. Using these orientations, state a definition of the linking number $lk(f^{-1}(x), f^{-1}(y))$.

(b) Suppose f is smoothly homotopic to a constant map. Show that in this case $lk(f^{-1}(x), f^{-1}(y)) = 0$. [Hint: you may use the fact that the linking number may be computed as the intersection number of surfaces bounded by the 1-manifolds in D^4.]

Question from Nick ... What sort of answer would one want in part (a)?