Topology General Exam
August 16, 2010

Name: ______________________________

Instructions: This is a four hour exam and 'closed book'. There are eight problems.

1. (a) Let $T \subset \mathbb{R}^5$ be a closed subspace homeomorphic to $\mathbb{R}^2$. Explain why T will be a retract of $\mathbb{R}^5$.

(b) View S^n as $\mathbb{R}^n \cup \{\infty\}$, so that the open subsets of S^n containing ∞ are precisely the complements of compact subsets of $\mathbb{R}^n$. Recall that a continuous function $f : \mathbb{R}^m \to \mathbb{R}^n$ is called *proper* if $f^{-1}(C)$ is compact in $\mathbb{R}^m$ whenever C is compact in $\mathbb{R}^n$. Show that such a proper map extends uniquely to a continuous function $\bar{f} : S^m \to S^n$.

(c) With T as in part (a), check that the inclusion $i : T \hookrightarrow \mathbb{R}^5$ is proper. By contrast, show that *no* retraction $r : \mathbb{R}^5 \to T$ can be proper. (Hint: start by using part (b).)

2. Let $\mathbb{R}^\infty$ denote the union $\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \ldots$, with the union topology, i.e. $U \subset \mathbb{R}^\infty$ is open iff $U \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for all n. Let $\mathbb{R}^\omega$ denote the product of a countable number of copies of $\mathbb{R}$, with the product topology. Check that the evident set theoretic inclusion $i : \mathbb{R}^\infty \to \mathbb{R}^\omega$ is continuous, but is *not* a homeomorphism onto its image.

3. (a) Describe a connected double cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$. (There is more than one correct answer.)

(b) What are the homology groups of your double cover?

(c) What is the fundamental group of your double cover?

4. Let M be the compact surface with boundary circle C as pictured:

(a) Explain why M is homotopy equivalent to a figure eight. (Hint: M is the torus with a disk removed, and the torus is often represented as a square with opposite edges identified.)

(b) Explain why the inclusion $i : C \hookrightarrow M$ induces the zero homomorphism from $H_1(C)$ to $H_1(M)$.

(c) By contrast, explain why i is *not* null homotopic.

5. Suppose given a commutative diagram of abelian groups

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & B_1 & \xrightarrow{\beta} & B_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_2 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with exact rows and columns. Show that there are isomorphisms

$$\ker\alpha \simeq \ker\beta \quad \text{and} \quad \operatorname{coker}\alpha \simeq \operatorname{coker}\beta.$$

6. Recall that an n–dimensional manifold is a Hausdorff topological space M that can be covered by open sets homeomorphic to open sets in $\mathbb{R}^n$. Prove that a compact n–dimensional manifold can be embedded in (i.e. is homeomorphic to a subset of) $\mathbb{R}^N$ for large enough N. (Hint: use a partition of unity associated to a finite open cover $U_1, \ldots, U_k$ of M equipped with embeddings $f_i : U_i \to \mathbb{R}^n$.)

7. Let $C \subset \mathbb{R}^3$ be the union of the x–axis and the y–axis. Compute $H_*(\mathbb{R}^3 - C)$. (Hint: note that $\mathbb{R}^3 - C = (\mathbb{R}^3 - x\text{-axis}) \cap (\mathbb{R}^3 - y\text{-axis})$.)

8. Let X be a Hausdorff space, and $f : X \to X$ a continuous function such that

- $f(x) \neq x$ for all $x \in X$, and
- $f \circ f$ is the identity.

(a) Show that every $x \in X$ has an open neighborhood W_x satisfying $f(W_x) \cap W_x = \emptyset$.

(b) Let $\bar{X} = X/(x \sim f(x))$, with the quotient space topology. Show that the quotient map $q : X \to \bar{X}$ is a covering map.