REAL ANALYSIS GENERAL EXAM FALL 2022

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems.

Problem 1.

Let (X, μ) be a σ-finite measure space and $p \in [1, +\infty)$. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $L^p(X, \mu)$ and suppose that $\|f_n\|_p \leq 1$, and that f_n converges a.e. to a measurable function f. Show that $\|f\|_p \leq 1$.

Problem 2.

Let μ be a Borel probability measure on $\mathbb{R}$ without atoms. Suppose that $E \subseteq \mathbb{R}$ is a Borel set with $\mu(E) > 0$. Show that there is a $t \in \mathbb{R}$ with $\mu(E \cap (-\infty, t)) = \frac{1}{2}\mu(E)$.

Problem 3.

Let X be a set equipped with a σ-algebra of sets Σ. Suppose that $\mu, \nu \colon \Sigma \to [0, +\infty)$ are finite measures. Set $\lambda = \mu + \nu$. Let $f \colon X \to \mathbb{R}$ be any Σ-measurable function so that

$$\nu(E) = \int_E f \, d\lambda$$

for all $E \in \Sigma$.

(i) Show that $0 \leq f \leq 1$ λ-a.e.

(ii) If $F = \{x : f(x) = 1\}$, show that $\mu(F) = 0$.

(iii) If $A \subseteq \{x : 0 \leq f(x) < 1\}$ and $\mu(A) = 0$, show that $\nu(A) = 0$.

Problem 4.

Fix $p \in [1, +\infty)$. Let $W^p([0, 1])$ consist of all absolutely continuous functions $f \colon [0, 1] \to \mathbb{C}$ so that $f' \in L^p([0, 1])$. For $f \in W^p([0, 1])$ define

$$\|f\| = |f(0)| + \|f'\|_p.$$

Show that $\| \cdot \|$ is a norm which makes $W^p([0, 1])$ into a Banach space. (You are allowed to use that $L^p([0, 1])$ is a Banach space).

Problem 5.

Let m be Lebesgue measure on $\mathbb{R}$. Let $\Omega = \{1_E : E \subseteq \mathbb{R} \text{ is Borel and } m(E) < +\infty\}$ regarded as a subset of $L^1(\mathbb{R})$ (recall that we identify two elements of L^1 if they agree almost everywhere). Throughout this problem regard Ω as a metric space equipped with the L^1-distance.

(i) If $a < b$ are real numbers, show that the function $\Omega \to \mathbb{R}$ given by

$$1_E \mapsto m(E \cap [a, b])$$

is a continuous function.

(ii) If $a < b$ are real numbers, let $U_{a,b}$ be the subset of Ω consiting of all 1_E where $E \subseteq \mathbb{R}$ is Borel and

$$0 < m(E \cap [a, b]) < b - a.$$

Show that $U_{a,b}$ is open and dense in Ω.

(iii) Let D be the set of all 1_E where $E \subseteq \mathbb{R}$ is Borel and so that for every interval I of positive measure we have

$$0 < m(E \cap I) < m(I).$$

Show that there is a countable collection $\{U_j\}_{j \in J}$ of open and dense subsets of Ω with

$$D \supseteq \bigcap_{j \in J} U_j.$$