Instructions. *2 hours. Closed book examination. Be neat in your presentation. When invoking a theorem from previous courses, name the theorem and thoroughly check its hypotheses. You must solve a significant portion of each of the three problems in order to pass the exam.*

---

1. Let $(X, \mathscr{M}, \mu)$ be a measure space. Let (f_n) be a sequence of nonnegative functions functions in $L^1(X, \mathscr{M}, \mu)$ and let f be a nonnegative function in $L^1(X, \mathscr{M}, \mu)$. Suppose that

$$\int_X f_n \, \mathrm{d}\mu \longrightarrow \int_X f \, \mathrm{d}\mu$$

and that $f_n \to f$ pointwise. Prove that f_n converges to f in $L^1(X, \mathscr{M}, \mu)$. Hint: consider $g_n = \min(f, f_n)$.

2. Let $(X, \mathscr{M}, \mu)$ be a measure space. Let $p \in [1, \infty)$.

(a) Let (f_n) be a sequence of functions in $L^p(X, \mathscr{M}, \mu)$ and let f be a function in $L^p(X, \mathscr{M}, \mu)$. Suppose that f_n converges to f in $L^p(X, \mathscr{M}, \mu)$. Prove that there exists a subsequence (f_{n_k}) such that for μ-almost all x, $\lim_{k\to\infty} f_{n_k}(x) = f(x)$. Hint: remember the proof of completeness of L^p.

(b) Let h be a measurable function on X. Let

$$D = \{f \in L^p(X, \mathscr{M}, \mu) \mid hf \in L^p(X, \mathscr{M}, \mu)\} .$$

Let (f_n) be a sequence of elements of D, and let $f, g \in L^p(X, \mathscr{M}, \mu)$ be such that f_n converges to f in L^p, and hf_n converges to g in L^p. Show that $f \in D$ and $g = hf$.

3. For μ a Borel probability measure on $\mathbb{R}$, we will denote by $\widehat{\mu}$ the function $\mathbb{R} \to \mathbb{C}$ given by

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{itx} \mathrm{d}\mu(x) .$$

We will also adopt the notational convention $\mathrm{sinc}(x) = \frac{\sin x}{x}$ if $x \neq 0$ and $\mathrm{sinc}(0) = 1$.

(a) Show that $\widehat{\mu}$ is a bounded continuous function.

(b) Let $\delta > 0$. Show that

$$\frac{1}{2\delta} \int_{-\delta}^{\delta} (1 - \mathrm{Re}(\widehat{\mu}(t))) \, \mathrm{d}t = \int_{\mathbb{R}} (1 - \mathrm{sinc}(\delta x)) \, \mathrm{d}\mu(x) .$$

(c) Show that for all $u \in \mathbb{R}$,

$$1 - \mathrm{sinc}(u) \geq \frac{1}{2} \, \chi_{(-\infty,-2)\cup(2,\infty)}(u) ,$$

and deduce that

$$\mu(\{x \in \mathbb{R} \mid |x| > 2\delta^{-1}\}) \ \leq \ \frac{1}{\delta}\int_{-\delta}^{\delta}(1 - \mathrm{Re}(\widehat{\mu}(t)))\ \mathrm{d}t\ .$$

(d) Let μ_n be a sequence of Borel probability measures on $\mathbb{R}$. Suppose that for all t, the limit $\Phi(t) = \lim_{n\to\infty}\widehat{\mu_n}(t)$ exists and that the resulting function $\Phi(t)$ is continuous at $t = 0$. Prove that for all $\epsilon > 0$, there exists a compact set K inside $\mathbb{R}$ such that, for all n, $\mu_n(K) \geq 1 - \epsilon$.