COMPLEX ANALYSIS GENERAL EXAM SPRING 2022

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems. Throughout $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Don't use any of the Picard theorems.

Problem 1.

Compute, for $0 < \alpha < 1$

$$\int_0^\infty \frac{1}{x^\alpha(1+x)}\,dx.$$

Show all estimates.

Problem 2.

Suppose that f is an entire function, and that there are constants $a, b > 0$ so that

$$|f(z)| \le a + b|z| \text{ for all } z \in \mathbb{C}.$$

Show that f is a polynomial of degree at most one.

Problem 3.

Give an explicit example of an *unbounded* harmonic function $u \colon \mathbb{D} \to (0, +\infty)$ with the property that

$$\lim_{z \to \zeta} u(z) = 0$$

for all $\zeta \in \partial\mathbb{D}$ with $\zeta \neq 1$. It is acceptable to leave your answer as the real (or imaginary) part of an explicit holomorphic function.

Problem 4.

Let $S = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1\}$. Suppose that $f \colon \overline{S} \to \mathbb{C}$ is bounded, continuous, and that $f\big|_S$ is analytic. If

$$\sup_{t \in \mathbb{R}} \max(|f(it)|, |f(1+it)|) \le 1,$$

show that $|f(z)| \le 1$ for all $z \in S$.

Suggestion: consider, for $\varepsilon > 0$, the function $f_\varepsilon(z) = \frac{f(z)}{1+\varepsilon z}$. Show that $|f_\varepsilon| \le 1$ for all $\varepsilon > 0$, and use this to conclude that $|f| \le 1$.

Problem 5.

Let $(M_n)_{n=0}^\infty$ be a sequence of positive real numbers. Assume that the series $\sum_{n=0}^\infty M_n z^n$ has radius of convergence at least 1. Let $\mathcal{F}$ be the set of holomorphic functions on $\mathbb{D}$ which satisfy

$$\left| \frac{f^{(n)}(0)}{n!} \right| \le M_n \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Show that $\mathcal{F}$ is normal (i.e. the closure of $\mathcal{F}$ is compact for the topology of uniform convergence on compact subsets of $\mathbb{D}$).

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