REAL ANALYSIS GENERAL EXAM FALL 2022

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems.

Problem 1.

Compute

$$\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx.$$

Problem 2.

Fix $a<b$ in $\mathbb{R}$. Recall that $h\colon[a,b]\to\mathbb{C}$ is *absolutely continuous* if for every $\varepsilon>0$ there is a $\delta>0$ so that if $((a_j,b_j))_{j=1}^k$ are disjoint intervals in $[a,b]$ with $\sum_{j=1}^k(b_j-a_j)<\delta$, then $\sum_{j=1}^k(f(b_j)-f(a_j))<\varepsilon$. For a Lipschitz function $g\colon[a,b]\to\mathbb{C}$ we set

$$\|g\|_{Lip}=\sup_{x\neq y,x,y\in[a,b]}\frac{|g(x)-g(y)|}{|x-y|}.$$

(a) Show that $f\colon[a,b]\to\mathbb{C}$ is Lipschitz if and only if f is absolutely continuous and $f'\in L^\infty([a,b])$.
(b) If $f\colon[a,b]\to\mathbb{C}$ is Lipschitz, show that $\|f\|_{Lip}=\|f'\|_\infty$.

Problem 3.

Let (X,μ) be a σ-finite measure space. Show that if $f,g\in L^1(X,\mu)$ wih $0\le f,g$ a.e., then

$$\|f-g\|_1=\int_0^\infty \mu(\{x:f(x)>t\}\Delta\{x:g(x)>t\})\,dt.$$

Here $E\Delta F=E\setminus F\cup F\setminus E$ for sets $E,F\subseteq X$. Suggestion: it might be helpful to first show that for $a,b\in[0,\infty)$ we have

$$|a-b|=\int_0^\infty |1_{(t,\infty)}(a)-1_{(t,\infty)}(b)|\,dt.$$

Note: for this problem you may take for granted that the function $X\times(0,\infty)\to\{0,1\}$ given by $(y,t)\mapsto 1_{\{x:f(x)>t\}}(y)$ and that the function $t\mapsto\mu(\{x:f(x)>t\}\Delta\{x:g(x)>t\})$ are measurable functions.

Problem 4.

Let (X,Σ) be a measurable space. Recall that if η is a signed measure on Σ, then $|\eta|=\eta_1+\eta_2$ where η_1,η_2 are the unique nonnegative measures with $\eta=\eta_1-\eta_2$ and $\eta_1\perp\eta_2$. Further, $\|\eta\|_{TV}=|\eta|(X)$. Suppose that μ,ν are signed measures on Σ, that $\|\mu\|_{TV},\|\nu\|_{TV}<+\infty$ and that $|\mu|,|\nu|$ are mutually singular.

(a) If $\mu=\mu_1-\mu_2$, $\nu=\nu_1-\nu_2$ with μ_i,ν_j nonnegative measures and $\mu_1\perp\mu_2$, $\nu_1\perp\nu_2$, show that $\mu_i\perp\nu_j$ for all $i,j\in\{1,2\}$.
(b) Show that

$$\|\mu+\nu\|_{TV}=\|\mu\|_{TV}+\|\nu\|_{TV}.$$

Problem 5.

(a) For $f \in L^1([0,1])$, set L_f be the set of $x \in [0,1]$ so that

$$\lim_{r\to 0} \frac{1}{2r} \int_{(x-r,x+r)} |f(y) - f(x)|\, dy = 0.$$

State the conclusion of the Lebesgue's differentiation theorem for L_f.

(b) For $n \in \mathbb{N}$, and $0 \le j \le 2^n - 1$, set $I_{n,j} = [j2^{-n}, (j+1)2^{-n})$. For $f \in L^1([0,1])$, define

$$E_n f = \sum_{j=0}^{2^n-1} \left(\frac{1}{m(I_{n,j})} \int_{I_{n,j}} f(t)\, dt \right) 1_{I_{n,j}}.$$

Show that

$$\lim_{n\to\infty} (E_n f)(x) = f(x) \text{ for almost every } x \in [0,1].$$