COMPLEX ANALYSIS GENERAL EXAM FALL 2022

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems. Throughout $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Don't use any of the Picard theorems.

Problem 1.

Compute, for $\xi > 0$,$b > 0$ and $a \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{(x-a)^2 + b}\, dx.$$

Show all estimates.

Problem 2.

Let f be an entire function with

$$\lim_{R\to\infty} \left(\sup_{|z|>R} \frac{|f(z)|}{|z|} \right) = 0.$$

Show that f is constant.

Problem 3.

Let $R > 0$, and $B_R(0) = \{z \in \mathbb{C} : |z| < R\}$. Suppose that $f \colon \mathbb{C} \setminus \overline{B_R(0)} \to \mathbb{C}$ is analytic and that $\lim_{z\to\infty} f(z) = 0$. Show that $\lim_{z\to\infty} zf(z)$ exists.

Hint: it maybe to helpful to consider the "singularity of f at ∞" namely, the signularity of $g(z) = f(1/z)$ at $z = 0$.

Problem 4.

Suppose that f_n is a sequence of entire functions and that f_n converges uniformly on compact subsets of $\mathbb{C}$ to a polynomial p of degree d. Show that there is an $N \in \mathbb{N}$ so that for all $n \geq N$, the function f_n has at least d zeroes (counted with multiplicity).

Problem 5.

Let U be open and connected. Recall that a collection $\mathcal{F}$ of analytic functions $U \to \mathbb{C}$ is *normal* if given any sequence $(f_n)_n$ in $\mathcal{F}$, there is a subsequence f_{n_k} which converges uniformly on compact sets. Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Fix $p \in U$ and let $\mathcal{F}$ be the collection of analytic functions $f \colon U \to \mathbb{H}$ so that $f(p) = i$. Show that $\mathcal{F}$ is *normal*.

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