Real analysis Qualifying exam, January 2021

DO NOT WRITE YOUR NAME ON YOUR WORK

My cellphone in case of zoom disconnection: ***

In order to receive the full credit for a problem, a detailed argument (rather than a sketch of the proof) is needed. Whenever applying one of the standard theorems, please indicate that clearly. Full solutions on a smaller number of problems will be worth more than partial solutions on more problems.

1. Let f_n, $n \geq 1$, and f be measurable functions on a space $(\Omega, \mathcal{F}, \mu)$, such that $f_n \to f$ in measure. Does this imply that there exists a measurable set $A \subseteq \Omega$ with $\mu(\Omega \setminus A) = 0$ such that $f_n(x) \to f(x)$ for all $x \in A$?
If yes, prove this. If no, give a counterexample.

2. Let B be a measurable subset of the two-dimensional plane such that the intersection of B with every vertical line is finite or countable. Find $\mu(B)$, where μ is the two-dimensional Lebesgue measure. Justify your answer.

3. Let $(\Omega, \mathcal{F})$ be a measurable space, and μ, ν, ρ be three finite positive measures on $(\Omega, \mathcal{F})$ such that $\mu \ll \nu$ (i.e., μ is absolutely continuous with respect to ν). Show that there exists a measurable function f on Ω such that for all $E \in \mathcal{F}$ we have

$$\mu(E) = \int_E f\, d\nu + \int_E (f-1)\, d\rho.$$

(Hint: use Radon-Nikodym's Theorem)

4. Let f, g be nonnegative measurable functions on $[0,1]$, and $a, b, c, d \geq 0$ be arbitrary nonnegative numbers. Show that then

$$\left(ac + bd + \int_0^1 f(x)g(x)\, dx\right)^3 \leq \left(a^3 + b^3 + \int_0^1 (f(x))^3\, dx\right)\left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2}\, dx\right)^2.$$

Partial credit is given for proving the inequality in the particular case $a = b = c = d = 0$.

5. Let $f(x)$ be a continuous function on $[0,1]$. Show that for every $\varepsilon > 0$ there exists $n \in \mathbb{Z}_{\geq 0}$ and constants $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that for the differential operator

$$D := \sum_{k=0}^{n} a_k \left(\frac{d}{dx}\right)^k = a_0 + a_1 \frac{d}{dx} + a_2 \left(\frac{d}{dx}\right)^2 + \ldots + a_n \left(\frac{d}{dx}\right)^n$$

we have $\left|f(x) - e^{x^2}(De^{-x^2})\right| < \varepsilon$ for all $x \in [0,1]$. Here $e^{x^2}(De^{-x^2})$ is the function obtained by applying D to e^{-x^2} and after that multiplying the result by e^{x^2}.