Complex Analysis General Exam Fall 2021

August 13, 2021

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems. You may assume earlier parts of a problem on later parts. E.g. if you solve part (b) of a problem assuming part (a), but cannot solve part (a), you will get full points for part (b). Throughout $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Don't use any of the Picard theorems.

Problem 1.

Compute, for $\xi > 0$,

$$\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{x^2 - 2x + 2} \, dx.$$

Show all estimates.

Problem 2.

Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is entire and that

$$\lim_{|z| \to +\infty} \frac{|f(z)|}{|z|} = 0,$$

show that f is constant.

Problem 3.

Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is entire and that there exists constants $R, C > 0$ so that $|f(z)| \geq C$ if $|z| \geq R$. Show that f is a polynomial.

Hint: it may be helpful to consider $g\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ *given by* $g(z) = f(1/z)$.

Problem 4.

Let $U \subseteq \mathbb{C}$ be a nonempty connected and open set. Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of holomorphic functions $f_n\colon U \to \mathbb{D}$ and that $(f_n)_n$ converges uniformly on compact sets to $f\colon U \to \mathbb{C}$. Show that if there is a $p \in U$ with $|f(p)| = 1$, then f is constant.

Problem 5.

Let $f_n\colon \mathbb{D} \to \mathbb{D}$ be a sequence of holomorphic functions such that $f_n \to 0$ pointwise on $\{z \in \mathbb{C} : |z| \leq 1/2\}$.

1. Suppose that $f\colon \mathbb{D} \to \mathbb{C}$ is a limit (uniformly on compact sets) of a subsequence of $(f_n)_n$. Show that $f = 0$.

2. Show that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. (You are allowed to use that there is a metric so that convergence with respect to that metric is the same as uniform convergence on compact sets).