Real analysis general exam, January 2020

1. Let μ be the Lebesgue measure on $\mathbb{R}$. For a Lebesgue measurable set $A \subset [0,1]$, is it true that

   (a) $\mu(A) = \sup_{U \subset A,\, U \text{ open}} \mu(U)$? If true, prove this. If false, give a counterexample.

   (b) $\mu(A) = \inf_{U \supset A,\, U \text{ open}} \mu(U)$? If true, prove this. If false, give a counterexample.

2. Find a polynomial $P(x)$ of degree at most 3 such that $\displaystyle\int_{-1}^{1} |x^4 - P(x)|^2 \, dx$ is minimal.

3. Let X be a compact metric space, and $C(X)$ be the space of all real-valued continuous functions on X with the supremum norm. Assume that the subset $\mathcal{A} \subset C(X)$ satisfies the following properties:

   - (algebra) For all $f, g \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$ we have $\alpha f + \beta g \in \mathcal{A}$ and $fg \in \mathcal{A}$.
   - (separates points) For any $x \neq y$ from X there exists a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

   This question has two parts:

   (a) Show by example that $\mathcal{A}$ need not be dense in $C(X)$, explicitly checking all the properties of your example $\mathcal{A}$.

   (b) In order to conclude that $\mathcal{A}$ is dense by Stone-Weierstrass Theorem, what additional condition(s) should be added?

4. Let μ be a measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra. Let $\mu(\mathbb{R}) = 1$. Next, let $\mathcal{F} \subset \mathcal{B}$ be the sub-σ-algebra of symmetric Borel sets, that is, $\mathcal{F}$ generated by all intervals of the form $(-a, a)$ with $a > 0$.

   Let $f \in L^1(\mathbb{R}, \mathcal{B}, \mu)$. Find a function g such that:

   (a) $g \in L^1(\mathbb{R}, \mathcal{F}, \mu)$ (in particular, g is $\mathcal{F}$-measurable).

   (b) For all $E \in \mathcal{F}$ we have $\displaystyle\int_E g \, d\mu = \int_E f \, d\mu$.

5. Let μ be a finite measure on some measurable space $(X, \mathcal{F})$.

   Show that a sequence of $\mathcal{F}$-measurable functions f_n converges to a function f in measure if and only if $\displaystyle\int_X \min\{1, |f_n - f|\} \, \mu(dx) \to 0$ as $n \to +\infty$.