Problem 1.

Compute

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\sin(x)}{x}\,dx$$

using Residue theory.

Problem 2.

Let f be a nonconstant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$. You are not allowed to use any of the Picard's theorem.

Problem 3.

Let f be an entire function. Assume there is a sequence $r_n \in (0,\infty)$ with $r_n \to \infty$, and constants $C, \alpha \in (0,\infty)$ with

$$\sup_{z\in\mathbb{C}:|z|=r_n}|f(z)| \le C r_n^{\alpha}.$$

Show that f is a polynomial.

Problem 4.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $(f_n)_n$ be a sequence of holomorphic functions and suppose that f_n converge uniformly on compact subsets of $\mathbb{D}$ to a function f. Suppose f has no zeroes on $\{z \in \mathbb{C} : |z| = \frac{1}{2}\}$. Show that there exists an $N \in \mathbb{N}$ so that if $n, m \ge N$, then f_n, f_m have the same number of zeroes in $\{z \in \mathbb{C} : |z| < 1/2\}$ (counting with multiplicity). (This is Hurwitz's theorem, you are not allowed to just quote this theorem).

Problem 5.

Let U be an open subset of $\mathbb{C}$. Let $f_n : U \to \mathbb{C}$ be a sequence of analytic functions. Suppose that f_n converge pointwise to a function $f : U \to \mathbb{C}$. If $|f_n| \le 1$ for every n, show that f is analytic.