Real analysis Qualifying exam, August 2020

Make sure that you have signed the Honor Pledge on Collab.

In order to receive the full credit for a problem, a detailed argument (rather than a sketch of the proof) is needed. Whenever applying one of the standard theorems, please indicate that clearly.

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, almost everywhere differentiable, and such that $f'(x) = 1$ almost everywhere. (Both "almost everywhere" properties are assumed with respect to the Lebesgue measure on $\mathbb{R}$.) Does this imply that $f(2) - f(1) = 1$?

If yes, prove this. If no, give a counterexample.

2. Is every open set in $\mathbb{R}^2$ a countable union of closed sets?

If yes, prove this. If no, give a counterexample.

3. Let $\mathcal{H}$ be a separable complex Hilbert space with basis (complete orthonormal system) $f_1, f_2, f_3, \ldots$. Define a linear operator P in $\mathcal{H}$ by setting

$$P(f_n) = f_{n+1}, \qquad n = 1, 2, \ldots.$$

(a) Find the adjoint P^* to P.

(b) Find the operators PP^* and P^*P.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Let $f_n\colon X \to \mathbb{R}$ be measurable functions such that for all $t \in \mathbb{R}$,

$$\lim_{n\to+\infty} \mu\,(x\colon f_n(x) \le t) = \begin{cases} 0, & t < 0; \\ 1, & t \ge 0. \end{cases}$$

Show that $f_n \to 0$ in measure.

5. Show that the operator

$$(Tf)(x) := \int_0^\infty \frac{f(y)}{x+y}\, dy$$

is bounded in the space $L^p(\mathbb{R}_{\ge 0})$ for all $1 < p < +\infty$.