Complex Analysis General Exam, August 2020

General guidelines: In order to receive full credit, a detailed argument (rather than a sketch of the proof) is needed. Whenever applying one of the standard theorems, please indicate that clearly.

Pledge:

1. Let $f$ be an entire function. Assume that

$$\max_{|z|=r} |f(z)| \leq 10 \log r$$

for $r \geq 100$. Shown that $f$ is constant.

2. Let $f$ be holomorphic on $\Omega = \{0 < |z| < 1\}$, and suppose that
   (i) $|f(z)| > 1$ for all $z \in \Omega$,
   (ii) there exists a sequence $\{z_k\}_{k=1}^{\infty} \subset \Omega$, $z_k \to 0, k \to \infty$ such that $|f(z_k)| \leq 10$ for $k \geq 100$.

Is 0 a removable singularity?

3. Compute

$$\int_0^\infty \frac{\log x}{x^2+1}\,dx$$

via complex integration. `Show all estimates`.

4. Let $f : \mathrm{D} = \{|z| < 1\} \to \mathbb{C}$ be holomorphic. Assume that $f(0) = 0$ and

$$|\mathrm{Re} f(z)| < 1, \text{ for all } \ z \in \mathrm{D}.$$

Show that

$$|f'(0)| \leq \frac{4}{\pi}.$$

Hint: A function $g(w) = \dfrac{e^{i\pi\frac{w}{2}} - 1}{e^{i\pi\frac{w}{2}} + 1}$ mapping $\{|\mathrm{Re}\, w| < 1\}$ conformally to D might prove useful.

5. Let $\mathcal{F}$ be a family of entire functions with the property that for any circle $C$, there exists a constant $M_C > 0$ such that

$$\sup_{f \in \mathcal{F}} \max_{z \in C} |f(z)| \leq M_C.$$

(i) show that there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ converging to an entire function $g$ uniformly on compacts in $\mathbb{C}$
(ii) suppose that $f_n(z) \neq 7$ for $n \geq 77$ and all $z \in \{|z| < 1\}$. Can $g\left(\frac{1+i}{2}\right) = 7$?