Real analysis Qualifying exam, August 2019

1. Let $\mathcal{C}$ be the Cantor set on $[0,1]$. Recall that it is obtained by iteratively deleting the open middle third: $(\frac{1}{3}, \frac{2}{3})$, then $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, and so on.

   (a) Show that $\mathcal{C} + \mathcal{C} := \{a + b \colon a, b \in \mathcal{C}\}$ is the full segment $[0, 2]$.

   (b) Find two sets $A, B \subset \mathbb{R}$, each of which is closed and has Lebesgue measure zero, such that $A + B = \{a + b \colon a \in A,\ b \in B\}$ is the full line $\mathbb{R}$.

2. Does there exist a measure space $(X, \mathcal{F}, \mu)$ with a finite measure μ, and a sequence of μ-measurable functions $\{f_n\}_{n=1,2,\ldots}$ on X such that:

   - $f_n(x) \geq 0$ for all n, x;
   - $f_n(x) \to 0$ as $n \to +\infty$ for all x;
   - $\int f_n(x)\mu(dx) \to 0$ as $n \to +\infty$;
   - $\Phi(x) := \sup_n f_n(x)$ has infinite integral?

   If yes, give an example of such a sequence $\{f_n\}$. If no, give a proof of nonexistence.

3. Let μ be a signed Borel measure on $\mathbb{R}^n$ which is bounded on bounded sets. Suppose that $\int f\, d\mu = 0$ for all continuous functions f with bounded support. Show that then $\mu = 0$.

4. Let $L^1(\mathbb{R})$ be the space of Lebesgue integrable functions on $\mathbb{R}$. For a positive function $f \in L^1(\mathbb{R})$ show that the function $\dfrac{1}{f(x)}$ does not belong to $L^1(\mathbb{R})$.

   (Hint: look at the function $1 = f^{1/2} f^{-1/2}$.)

5. Applying the Gram-Schmidt orthogonalization to $1, x, x^2, \ldots$ in the Hilbert space $L^2([-1,1])$ (with Lebesgue measure), one gets the Legendre polynomials $L_n(x)$, $n = 0, 1, 2, \ldots$.

   (a) Show that the Legendre polynomials form a basis (= complete orthogonal system) in the Hilbert space $L^2([-1,1])$

   (b) Show that the Legendre polynomials are given by the formula $L_n(x) = c_n \frac{d^n}{dx^n}(x^2 - 1)^n$ (you do not need to specify c_n).

   (Hint: employ integration by parts.)