1. Let $C$ be the Cantor set on $[0, 1]$. Recall that it is obtained by iteratively deleting the open middle third: $(\frac{1}{3}, \frac{2}{3})$, then $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, and so on.

(a) Show that $C + C := \{a + b: a, b \in C\}$ is the full segment $[0, 2]$.

(b) Find two sets $A, B \subset \mathbb{R}$, each of which is closed and has Lebesgue measure zero, such that $A + B = \{a + b: a \in A, b \in B\}$ is the full line $\mathbb{R}$.

2. Does there exist a measure space $(X, \mathcal{F}, \mu)$ with a finite measure $\mu$, and a sequence of $\mu$-measurable functions $\{f_n\}_{n=1,2,...}$ on $X$ such that:

- $f_n(x) \geq 0$ for all $n, x$;
- $f_n(x) \to 0$ as $n \to +\infty$ for all $x$;
- $\int f_n(x)\mu(dx) \to 0$ as $n \to +\infty$;
- $\Phi(x) := \sup_x f_n(x)$ has infinite integral?

If yes, give an example of such a sequence $\{f_n\}$. If no, give a proof of nonexistence.

3. Let $\mu$ be a signed Borel measure on $\mathbb{R}^n$ which is bounded on bounded sets. Suppose that $\int f\,d\mu = 0$ for all continuous functions $f$ with bounded support. Show that then $\mu = 0$.

4. Let $L^1(\mathbb{R})$ be the space of Lebesgue integrable functions on $\mathbb{R}$. For a positive function $f \in L^1(\mathbb{R})$ show that the function $\frac{1}{f(x)}$ does not belong to $L^1(\mathbb{R})$.

(Hint: look at the function $1 = f^{1/2}f^{-1/2}$.)

5. Applying the Gram-Schmidt orthogonalization to $1, x, x^2, \ldots$ in the Hilbert space $L^2([-1, 1])$ (with Lebesgue measure), one gets the Legendre polynomials $L_n(x), n = 0, 1, 2, \ldots$.

(a) Show that the Legendre polynomials form a basis (= complete orthogonal system) in the Hilbert space $L^2([-1, 1])$

(b) Show that the Legendre polynomials are given by the formula $L_n(x) = c_n \frac{d^n}{dx^n}(x^2 - 1)^n$

(you do not need to specify $c_n$).

(Hint: employ integration by parts.)