Problem 1.

Let ξ be a nonnegative real number. Compute $\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{x^2+1}\, dx$.

Problem 2.

Let f be an entire function. Suppose that there is an $\alpha \in (0,\infty)$ and a $C > 0$ so that $|f(z)| \leq C|z|^\alpha$ for all $|z| \geq 1$. Show that f is a polynomial.

Problem 3.

Let f be an entire function. Suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

Problem 4.

Let $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Suppose that $f\colon \overline{\Omega} \to \mathbb{C}$ is continuous, and that $f\big|_\Omega$ is holomorphic. Suppose that $|f(iy)| \leq 1$ for all $y \in \mathbb{R}$ and $|f(z)| \leq 2$ for all $z \in \Omega$. Show that in fact $|f(z)| \leq 1$ for all $z \in \Omega$.

Hint: For $\varepsilon > 0$, consider $f_\varepsilon(z) = \frac{f(z)}{1+\varepsilon z}$. Show that $|f_\varepsilon| \leq 1$ for every $\varepsilon > 0$.

Problem 5.

Recall that if U is a open subset of $\mathbb{C}$ and $\mathcal{G}$ is family of holomorphic functions on U then we say that $\mathcal{G}$ is *normal*, if given any sequence $(f_n)_n$ in $\mathcal{G}$ there is a subsequence $(f_{n_k})_k$ and a holomorphic function $g\colon U \to \mathbb{C}$ with $f_{n_k} \to_{k\to\infty} g$ uniformly on compact subsets of U.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $\mathcal{F}$ is a family of holmorphic functions on $\mathbb{D}$ and that $\sup_{f\in\mathcal{F}} |f(0)| < \infty$. Show that $\mathcal{F}$ is normal if and only if $\{f' : f \in \mathcal{F}\}$ is normal.