NAME: ______________________________

PLEDGE: ____

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SIGNATURE: ________________________________

To get credit for a problem, you must include all logical steps and detailed calculations. Verify or give adequate reasons for assertions that you make. Cite by name any theorems you wish to invoke. Please write only on one side of your paper.
If a problem has multiple parts and it helps, you may, for example, use part (a) to prove part (b), even if you haven't proved part (a).

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PART I

1. Let f be an entire function such that $|f(z)| \leq \log(10 + |z|)$ for all z in $\mathbb{C}$. Show that f is a constant function.

2. Let f be analytic in a neighborhood of the closed disc $\overline{D(z_0, r)}$ and $0 < s < r$. Show that
$$|f(z)| \leq \frac{1}{\pi(r-s)^2} \iint_{D(z_0,r)} |f(x+iy)|\, dx\, dy$$
for all z in $\overline{D(z_0, s)}$.

3. Let Ω be an open set in $\mathbb{C}$ and $\mathcal{F}$ a family of analytic functions on Ω with the property that there exists a positive constant M such that
$$\iint_{\Omega} |f(x+iy)|\, dx\, dy \leq M$$
for all f in $\mathcal{F}$. Show that $\mathcal{F}$ is a *normal family*, i.e., there exists a sequence $\{f_n\} \subset \mathcal{F}$ such that $\{f_n\}$ converges uniformly on compacts in Ω (the inequality in problem 2. should be useful).

4. Show that
$$\int_0^\infty \frac{\log x}{x^2+1}\, dx = 0$$
(it may prove convenient to consider the branch of the logarithm corresponding to the cut along the 'negative' imaginary semi-axis).

5. Let
$$A = \{z \in \mathbb{C} : \, 0 < |z| < 1\} \quad \text{and} \quad B = \{z \in \mathbb{C} : \, 1 < |z| < 2\}.$$
Show that A and B are not conformally equivalent.

PART II

6. Let $\{E_i\}_{i=1}^{\infty}$ be a partition of a set X (that is, $E_i \cap E_j = \emptyset$ and $\cup_{j=1}^{\infty} E_j = X$). For any $J \subseteq \mathbb{N} = \{1, 2, \ldots\}$ let $E_J := \cup_{i \in J} E_i$ with the convention that $E_\emptyset = \emptyset$.
(a) Show that $\mathcal{M} := \{E_J\}_{J \subset \mathbb{N}}$ is the σ-algebra generated by $\{E_i\}_{i=1}^{\infty}$ (that is, the smallest σ-algebra containing $\{E_i\}_{i=1}^{\infty}$).
(b) Show that $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable iff f is constant on each set E_i.

7. Let f be a non-negative measurable function on $(0, 1)$. Suppose there is a constant c so that
$$\int_0^1 f^n(x)\, dx = c \text{ for all } n = 1, 2, 3, \ldots.$$
Show that there is a measurable set $A \subset (0, 1)$ such that $f(x) = 1_A(x)$ for a.e. $x \in (0, 1)$.

8. Suppose μ and ν are finite (positive) measures on a measurable space $(X, \mathcal{M})$. Show that there exists a measurable $f : X \to [0, \infty)$ such that for all $A \in \mathcal{M}$
$$\int_A (1 - f)\, d\mu = \int_A f\, d\nu$$

9. Recall that, for a function $f : \mathbb{R} \to \mathbb{C}$, we say f is *Lipschitz* if there exists $M \in (0, \infty)$ such that $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$.
Show that $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz iff f is absolutely continuous and $|f'| \le M$.

10. Let $X = [-\pi, \pi]$ and λ denote uniform measure on X, that is, $d\lambda(x) = \frac{1}{2\pi} dx$. Then $H := L^2(X, \lambda)$ equipped with the inner product
$$\langle u, v\rangle := \int_{-\pi}^{\pi} u(x)\overline{v}(x)\, d\lambda(x)$$
is a Hilbert space, and, for $e_k(x) := e^{ikx}$, $\{e_k\}_{k \in \mathbb{Z}}$ is an orthonormal basis of H.
Suppose that $u \in C^1(\mathbb{R}, \mathbb{C})$ is 2π-periodic, that is,
$$u(x + 2\pi) = u(x) \text{ for all } x \in \mathbb{R}.$$
(a) Show that $\langle u', e_k\rangle = ik\langle u, e_k\rangle$ for all $k \in \mathbb{Z}$, where $u'(x) = \frac{du}{dx}(x)$.
(b) Show that
$$s(x) := \sum_{k \in \mathbb{Z}} \langle u, e_k\rangle e^{ikx}$$
is convergent for all $x \in \mathbb{R}$ and that s is continuous.
(c) Show that $u(x) = s(x)$ for **all** $x \in \mathbb{R}$.