Instructions. *4 hours. Closed book examination. As the exam is a bit long you do not have to do everything in order to succeed. In order to pass, you need to complete a significant portion of each of the two sections: complex analysis (Questions 1 through 4) and real analysis (Questions 5 through 7). Fully completing one section while leaving the other untouched will not result in a passing grade so please plan the allocation of your time and effort accordingly.*

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1. Let f be an entire function. Which of the following conditions imply that f is constant? Give proofs or counterexamples.
   (a) $\operatorname{Re} z \geq 0 \Rightarrow |f(z)| \leq 1$.
   (b) $|z| \geq 1 \Rightarrow \left|\frac{f(z)^2}{z}\right| \leq \pi$.

2. Evaluate
$$\int_0^\infty \frac{1}{x^n+1}\,dx,$$
where $n \geq 2$ is an integer. Your final answer should not involve any reference to complex numbers.

3. (a) Let f be an analytic function. Show that the level curves $\operatorname{Re} f(z) = k_1$, $\operatorname{Im} f(z) = k_2$ are perpendicular at any point where $f'(z) \neq 0$.
   (b) Let $p_1, p_2, \ldots p_n$ be points on a circle of radius 1. Show that there is a point on the circle such that the product of its distances to the p_j is 1. (Suggestion: apply the maximum modulus principle to an appropriate complex polynomial.)

4. Let z_1, z_2 be unequal numbers in the open unit disk $\mathbb{D}$, and let f be an analytic self-map of $\mathbb{D}$, not necessarily 1-1. Follow the steps below to prove the following version of Pick's Lemma:
$$\left|\frac{f(z_1)-f(z_2)}{1-f(z_1)\overline{f(z_2)}}\right| \leq \left|\frac{z_1-z_2}{1-z_1\overline{z_2}}\right|.$$
   (a) Write down an explicit conformal self-map of the disk that sends $f(z_2)$ to 0. You do not need to prove anything about your map.
   (b) Find conformal self-maps of the disk φ, ψ such that $\varphi \circ f \circ \psi^{-1}$ is an analytic self-map of the disk sending 0 to 0, and apply the Schwarz lemma appropriately to $\varphi \circ f \circ \psi^{-1}$.

5. (a) Recall that continuous functions in $L^2(\mathbb{R})$ are dense in $L^2(\mathbb{R})$. Show that compactly supported continuous functions are dense in $L^2(\mathbb{R})$.

(b) For positive a and b let $f_{a,b} = a\chi_{[0,b]}$. Produce a sequence (a_n, b_n) such that $\int_{\mathbb{R}} f_{a_n,b_n}(x)\,dx = 1$ for all n while $||f_{a_n,b_n}||_{L^2} \to 0$ when $n \to \infty$.

(c) Show that compactly supported continuous functions f such that $\int_{\mathbb{R}} f(x)\,dx = 0$ are dense in $L^2(\mathbb{R})$.

(d) Are continuous functions f such that $\int_{[0,1]} f(x)\,dx = 0$ dense in $L^2([0,1])$? Justify your answer.

6. (a) Let $h_{0,0}(x) = \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1)}(x)$. More generally, for $n \geq 0$ and $0 \leq k \leq 2^n - 1$, we define $h_{n,k}(x) = 2^{\frac{n}{2}} h_{0,0}(2^n x - k)$. For $n \geq 0$ and $(x, y) \in [0,1]^2$ we let

$$L_n(x,y) = \sum_{k=0}^{2^n-1} h_{n,k}(x) h_{n,k}(y)$$

and

$$K_n(x,y) = \chi_{[0,1)}(x)\chi_{[0,1)}(y) + \sum_{m=0}^{n} L_m(x,y)\ .$$

Plot L_0, L_1, L_2 and K_0, K_1, K_2. (Do not use 3d plots but simply subdivide the square into regions and in each region indicate the value of the function. Do six different plot sketches, one for each function).

(b) Show that for $n \geq 0$ and $(x, y) \in [0,1]^2$, one has $K_n(x, y) = 2^{n+1}$ if x, y both belong to $\left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right)$ for some k, $0 \leq k \leq 2^{n+1} - 1$, and $K_n(x, y) = 0$ otherwise.

(c) Let f be continuous on $[0,1]$ and, for $n \geq 0$, define

$$g_n(x) = \int_0^1 K_n(x,y) f(y)\,dy\ .$$

Show that $\lim_{n\to\infty} g_n = f$ in $L^p([0,1])$ for all $p \in [1,\infty)$.

(d) Let V be the linear span of the collection of functions $h_{n,k}$, with $n \geq 0$, $0 \leq k \leq 2^n - 1$, together with the constant function equal to 1. Prove that V is dense in $L^p([0,1])$ for all $p \in [1,\infty)$.

(e) In the particular case $p = 2$, give a geometric interpretation for the map which produces g_n out of f.

7. (a) In this problem the formula

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^n\, e^{-\frac{x^2}{2}}\,dx = 1$$

for $n = 0$ and $n = 2$ can be used without justification. For $(t, x) \in (0,\infty) \times \mathbb{R}$ we let

$$P(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}\ .$$

Show that this function is infinitely differentiable and that

$$\frac{\partial^{m+n}P}{\partial t^m \partial x^n}(t,x) = \frac{Q_{m,n}(t,x)}{2\sqrt{\pi}\ t^{2m+n+\frac{1}{2}}} e^{-\frac{x^2}{4t}}$$

for some two-variable polynomials $Q_{m,n}(t,x)$ satisfying the recursion

$$\begin{aligned} Q_{m+1,n} &= t^2 \tfrac{\partial Q_{m,n}}{\partial t} + \left(\tfrac{x^2}{4} - \left(2m+n+\tfrac{1}{2}\right) t \right) Q_{m,n} \ , \\ Q_{m,n+1} &= t \tfrac{\partial Q_{m,n}}{\partial x} - \tfrac{x}{2} Q_{m,n} \ . \end{aligned}$$

(b) Let f be a compactly supported twice continuously differentiable function on $\mathbb{R}$. Show that

$$\psi(t,x) = \int_{\mathbb{R}} P(t, x-y)\ f(y)\ dy$$

is well defined and infinitely differentiable on $(0,\infty)\times\mathbb{R}$.

(c) By computing $Q_{0,0}, Q_{1,0}, Q_{0,1}, Q_{0,2}$ show that ψ satisfies the heat equation, i.e.,

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

on $(0,\infty)\times\mathbb{R}$.

(d) Recall that a twice continuously differentiable function f satisfies the Taylor formula with integral remainder

$$f(y) = f(x) + (y-x)f'(x) + (y-x)^2 \int_0^1 (1-s) f''(x+s(y-x))\ ds \ .$$

Extend ψ to $[0,\infty)\times\mathbb{R}$ by letting $\psi(0,x) = f(x)$ for all x. Show that the heat equation continues to hold for this extension with $\frac{\partial}{\partial t}$ derivatives at $t=0$ understood as right-derivatives. Hint: use the previous Taylor formula and a suitable change of variable in order to establish the identity

$$\frac{\psi(t,x)-\psi(0,x)}{t} = \frac{1}{2\sqrt{\pi}} \int_{[0,1]} \int_{\mathbb{R}} z^2 e^{-\frac{z^2}{4}} (1-s) f''(x+s\sqrt{t}\ z)\ dz\ ds$$

for $t > 0$.