Analysis General Exam: August 2015

Please present your solutions as proofs, including all logical steps and detailed calculations. Verify or give adequate reasons for assertions that you make. Cite by name any theorems you wish to invoke. Please write only on one side of your paper.

PART I

1. Prove that $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges to an analytic function in the half plane $H = \{z : \operatorname{Re} z > 1\}$. Show that the derivatives of $\zeta(z)$ converge uniformly on compact subsets of H.

2. Suppose D is a bounded domain with piecewise smooth boundary. Let $f(z)$ be meromorphic and $g(z)$ analytic on D. Suppose that both $f(z)$ and $g(z)$ extend analytically across the boundary of D, and that $f(z) \neq 0$ on ∂D. Show that

$$\frac{1}{2\pi i} \oint_{\partial D} g(z) \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{n} m_j g(z_j),$$

   where $z_1, \ldots, z_n$ are the zeros and poles of $f(z)$, and m_j is the order of $f(z)$ at z_j.

3. Recall that the principal branch of the inverse tangent function is defined on the complex plane with two slits on the imaginary axis by

$$\operatorname{Tan}^{-1} z = \frac{1}{2i} \operatorname{Log}\left(\frac{1+iz}{1-iz}\right), \; z \notin (-i\infty, -i] \cup [i, i\infty).$$

   Find the derivative of $\operatorname{Tan}^{-1} z$.

   Consider the analytic continuation of $\operatorname{Tan}^{-1} z$ along the path $z(t) = \sqrt{3}e^{it}, t \in [0, 2\pi]$. What is the analytic continuation of $\operatorname{Tan}^{-1} z$ at the end of the path?

   Consider now the analytic continuation along a figure-eight path with the same starting and ending point as above, that circles i once in a counterclockwise direction and $-i$ once in a clockwise direction. What is the analytic continuation of $\operatorname{Tan}^{-1} z$ at the end of the path?

4. Consider the function

$$f(z) = \left(\frac{1+z}{1-z}\right)^2.$$

   Is f one-to-one on the unit disk $D = \{z : |z| < 1\}$? What is the image of D under f?

5. Compute

$$PV \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4)(x-1)} dx.$$

PART II

The functions below (in #6-9) are defined on the measure space $(X, \mathcal{M}, \mu)$, where μ is a **finite** measure. Don't forget to verify hypotheses in any theorems you use.

6. Show that if $\{f_n\}$ are measurable real-valued functions, then $g = \limsup f_n$ is also measurable. (Hint: find an expression for the set where $g > \alpha$, for any real α.)

7. Let $1 < p < \infty$. State Hölder's Inequality and use it to show that $L^p \subseteq L^1$. Show that this inclusion is proper if $\mathcal{M}$ contains sets of arbitrarily small positive measure.

8. Let $f \in L^\infty$. Show that $f \in L^p$ for any $p > 0$, and prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

9. Suppose $\{f_n\}$ and f are L^1 functions such that $f_n \to f$ a.e. Show that it need not be true that $f_n \to f$ in L^1, but this does follow if in addition $\|f_n\|_1 \to \|f\|_1$. (Hint: apply Fatou's Lemma to the functions $|f| + |f_n| \pm |f - f_n|$.)

10. Let $\mathfrak{X}$ be a real Banach space, with $\mathfrak{Y}$ a proper closed subspace.

    a) Define the quotient norm on $\mathfrak{X}/\mathfrak{Y}$ and show that it is complete, so that $\mathfrak{X}/\mathfrak{Y}$ is a Banach space. (You do not need to check that this is a norm.)

    b) Let $x \in \mathfrak{X} \setminus \mathfrak{Y}$. Show that there is $f \in \mathfrak{X}^*$ such that $f|_{\mathfrak{Y}} = 0$ and $f(x) \neq 0$. If $\|f\| = 1$, what is the greatest possible value for $f(x)$?