Analysis General Exam August 21, 2013

Name: ______________________________

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Instructions. *4 hours. To get credit for a problem, you must carefully justify all (nontrivial) claims and show all calculations. You may use without proof anything that is proved in the texts by Folland and Bak and Newman, or other standard reference. If you do so, either refer to the theorem by name (if it has one) or give its statement; also verify explicitly all of its hypotheses. You may not cite a statement you are explicitly asked to prove, or facts that were given as exercises or homework.*

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1. Find

$$\int_{\gamma_R} \frac{z}{\sqrt{(z-a)(z-b)}}dz$$

Here γ_R is the circle of radius R traversed counter-clockwise and $R > \text{Max}(|a|, |b|)$. The square root is defined to be continuous for $|z| > \text{Max}(|a|, |b|)$ and such that the integrand has limit 1 as $|z| \to \infty$.

2. Suppose f is an entire function satisfying

$$|f(z)| \leq C(|z|^3 + 1)/\ln(|z| + 2)$$

Show that f is a polynomial of degree at most two. You can assume the validity of Cauchy's integral formula for derivatives.

3. (a) Given $a \in D := \{z : |z| < 1\}$, show that there is an analytic function $f : D \to D$ such that

$$|f''(a)| = \text{Max}\{|g''(a)| : g \text{ is analytic in } D \text{ and } g : D \to D\}.$$

Make sure you check that $f : D \to D$?

(b) Suppose that the second derivative in (a) is replaced by the first (i.e. $f'' \to f'$ and $g'' \to g'$). Show that the function f would then satisfy $f(a) = 0$. You may want to make use of the function

$$g(z) = \frac{f(z) - f(a)}{1 - f(z)\bar{f}(a)}.$$

4. Consider the polynomial

$$P(z) = z^{84} + 17z^{54} + 68z^4 + z^3 - z + 1.$$

(a) How many zeros does P have in the disk of radius 1?

(b) How many zeros does P have in the disk of radius 2?

5. Consider the functions f and g defined by

$$f(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}\, dt \qquad \text{and} \qquad g(x) = \left(\int_0^x e^{-t^2}\, dt\right)^2 .$$

   (a) Prove that f is differentiable with continuous derivative and show that the latter satisfies $f' = -g'$.

   (b) By a careful comparison of the values of $f(x) + g(x)$ at 0 and ∞ prove that

$$\int_0^\infty e^{-t^2}\, dt \;=\; \frac{\sqrt{\pi}}{2} .$$

6. In this problem we consider Euler's Gamma Function defined for $x \in (0, \infty)$ by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$.

   (a) Use the change of variable $t = x(1+u)$ in order to express $\Gamma(x+1)$ in terms of the integral $\int_{-1}^\infty e^{-x\phi(u)}\, du$ where $\phi(u) = u - \log(1+u)$. (The only logarithm we use is the Neperian one)

   (b) Show there is an $\epsilon \in (0,1)$ such that $\phi(u) \geq \frac{u^2}{4}$ for $u \in [-\epsilon, \epsilon]$. For that fixed ϵ, and using the formula derived at the end of the previous problem, prove that

$$\lim_{x\to\infty} \int_{-\epsilon\sqrt{x}}^{\epsilon\sqrt{x}} e^{-x\phi\left(\frac{z}{\sqrt{x}}\right)}\, dz \;=\; \sqrt{2\pi} .$$

   (c) For the same fixed ϵ, show that $\int_{-1}^{-\epsilon} e^{-x\phi(u)}\, du$ and $\int_\epsilon^\infty e^{-x\phi(u)}\, du$ satisfy bounds of the form Ae^{-cx} for suitable positive constants A and c. (Convexity helps)

   (d) From the previous considerations deduce Stirling's asymptotic formula

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\; (1 + o(1))$$

   where $o(1)$ means some function of x that goes to zero when $x \to \infty$.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a measurable function.

   (a) Show that the map $[0,\infty) \to [0,\infty]$ given by $t \mapsto \mu(\{x | f(x) > t\})$ is measurable.

   (b) By first considering the case where f is a simple function and then using monotone convergence prove that

$$\int_X f\, d\mu \;=\; \int_{[0,\infty)} \mu\left(\{x \in X | f(x) > t\}\right)\, dm(t) . \qquad (*)$$

(c) Using a suitable countable decomposition, show that

$$\{(x,t) \in X \times [0,\infty) | f(x) > t\} \in \mathcal{M} \otimes \mathcal{B}_{[0,\infty)} \ .$$

Use this fact to derive an alternate proof of $(*)$ when X is σ-finite.

8. Let $(X, \mathcal{M}, \mu)$ be a σ-**finite** measure space and consider the real Hilbert space $\mathcal{H} = L^2(X)$ of real-valued square-integrable functions. Let f be a measurable function $X \to \mathbb{R}$ with finite L^∞ norm.

   (a) Show that the map $T_f : \mathcal{H} \to \mathcal{H}$ defined by pointwise multiplication $g \mapsto fg$ is well-defined, linear continuous and satisfies $||T_f|| = ||f||_\infty$.

   (b) A number $\lambda \in \mathbb{R}$ is called an eigenvalue of T_f iff there exists $g \neq 0$ in $\mathcal{H}$ such that $T_f(g) = \lambda g$. Show that λ is an eigenvalue of T_f iff $\mu(X_\lambda) > 0$ where $X_\lambda = \{x \in X | f(x) = \lambda\}$.

   (c) Show that the set of eigenvalues for T_f is at most countable.