ANALYSIS GENERAL EXAM — AUGUST 20, 2012

NAME: ______________________________

PLEDGE: ____

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SIGNATURE: ______________________________

Instructions. *4 hours. To get credit for a problem, you must carefully justify all (nontrivial) claims and show all calculations. You may use without proof anything that is proved in the texts by Folland and Bak and Newman, or other standard reference. If you do so, either refer to the theorem by name (if it has one) or give its statement; also verify explicitly all of its hypotheses. You may not cite a statement you are explicitly asked to prove, or facts that were given as exercises or homework.*

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Total points: 200

1. (30 points)

   (a) What very relevant property does an entire function f have if f satisfies

   $$|f(z)| \leq c(|z|+1)^{1/2}?$$

   Give a proof.

   (b) What very relevant property does a function f have if f is analytic in an open connected set $U \subset \mathbb{C}$ and

   $$f : U \to \{z : |z| = 1\}?$$

   Give a proof.

   (c) What very relevant property does a function u have if u is harmonic in $\mathbb{R}^2$ and $u(x,y) \geq 0$ for all x, y?

   Give a proof.

2. (25 points) Use the monotone class theorem to show that Lebesgue measure on $\mathbb{R}^d$ is outer regular. (Recall that μ is *outer regular* if for all Borel sets B, $\mu(B) = \inf\{\mu(U) : B \subset U \text{ open}\}$.)

3. (25 points) Suppose $p, q, r > 1$ satisfy $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$, and $(x_1, x_2, \ldots), (y_1, y_2, \ldots), (z_1, z_2, \ldots)$ are real sequences. Prove that

   $$\sum_{i=1}^{\infty} x_i y_i z_i \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q} \left(\sum_{i=1}^{\infty} |z_i|^r\right)^{1/r}.$$

4. (25 points) Show that the polynomial $2z^5 + 6z - 1$ has one root in $(0,1)$ and four roots in the annulus $\{z : 1 < |z| < 2\}$.

5. (20 points) Find

   $$\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2(\theta)} = \int_0^{2\pi} \frac{a\,d\phi}{2a^2 + 1 - \cos(\phi)}, \quad a > 0. \qquad (1)$$

6. (25 points) Suppose f is an entire function and g is analytic in

   $$\{z : |Im(z)| > 0\} \cup \{z : Im(z) = 0, Re(z) \in (-1,1)\}$$

   and in fact for $Im(z) > 0$,

   $$g(z) = \int_{-1}^{1} \frac{f(x)dx}{(x-z)}.$$

Show that for $Im(z) < 0$,

$$g(z) = \int_{-1}^{1} \frac{f(x)dx}{(x-z)} + 2\pi i f(z).$$

7. (40 points) For each of the following, determine if the statement is true (always) or false (not always true). If true, give a brief proof; if false, give a counterexample or prove false in some other rigorous way. No credit if reason or counterexample is wrong.
   (a) For $p > 1$, any bounded sequence in L^p has a convergent subsequence.
   (b) There exists a sequence of functions $f_n \in L^1([0,1])$ such that $f_n \to 0$ in L^1, but there is no subsequence f_{n_k} with $f_{n_k} \to 0$ pointwise a.e.
   (c) The space $C([0,1])$ is dense in $L^\infty([0,1])$.
   (d) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then its graph $G(f) = \{(x, f(x)) : x \in \mathbb{R}\}$ is a null set in $\mathbb{R}^2$.

8. (10 points) Suppose f_n is a sequence of measurable functions on the measure space $(X, \mathcal{M}, \mu)$. Assume that $f_n \to f$ μ-a.e. and there exists an integrable function F such that $|f_n| \le F$ μ-a.e. for each n. Show that $f_n \to f$ in measure.