GENERAL EXAM - ANALYSIS
August, 2011

Closed book, closed notes. Please pledge. In each problem, justify all assertions, show calculations, and identify those theorems which you invoke in your arguments.

1.

(a) Suppose that f is analytic in an open set containing the closed disk $\{z : |z| \leq R\}$ and that a, b are two complex numbers with $|a| < R$ and $|b| < R$. Evaluate

$$\int_{\gamma_R} \frac{f(z)}{(z-a)(z-b)}\, dz,$$

where γ_R is the positively oriented circle centered at 0 with radius R.

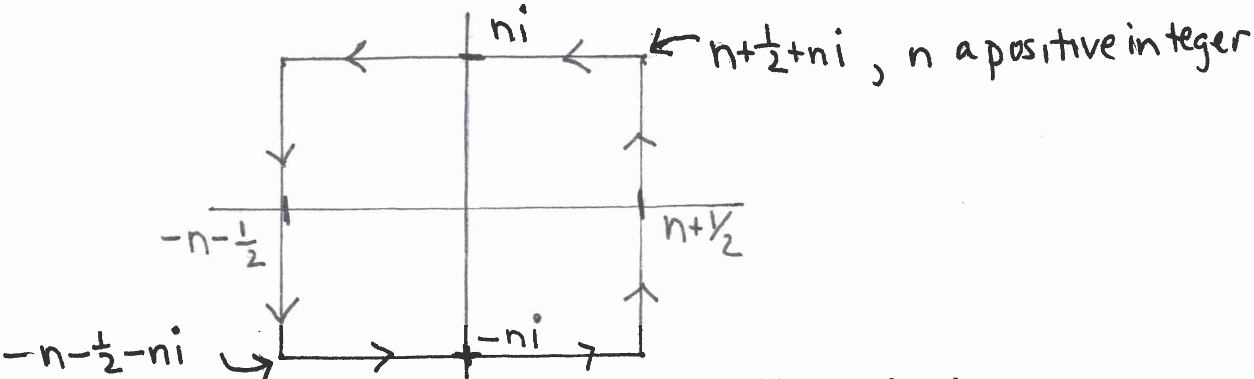
(b) Using your work in (a), prove Liouville's theorem on bounded entire functions.

2. Let a be a real number that is not an integer and let

$$f(z) = \frac{\pi \cos \pi z}{(z+a)^2 \sin \pi z}.$$

(a) Compute the residue of f at each of its singularities.

(b) Consider the rectangle γ_n as shown. Show that $\int_{\gamma_n} f(z)dz \to 0$ as $n \to \infty$. (You may give "order of magnitude" estimates in doing this.)



(c) Using the Residue Theorem and your work in (a) and (b), give a formula for

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^2}.$$

(d) What is

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2}?$$

3. Suppose that $a_1, a_2, \cdots, a_n$ are n points in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. Set

$$F(z) = \prod_{m=1}^{n} \frac{z - a_m}{1 - \overline{a_m} z}.$$

Show that for each $c \in \mathbb{D}$, the equation

$$F(z) = c$$

has n roots in $\mathbb{D}$ (counting multiplicities).

4. A function $f(z)$ that is analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is said to be *subordinate* to the analytic function $F(z)$ if $f(z) = F(g(z))$ for some function $g(z)$ that is analytic in $\mathbb{D}$ and satisfies $|g(z)| \leq |z|$ there. This is written $f \prec F$.

(a) Show that if $f \prec F$, then $|f'(0)| \leq |F'(0)|$.

(b) Suppose that f is analytic in $\mathbb{D}$, $f(0) = 0$ and $|\mathrm{Re}\, f(z)| < 1$ for all $z \in \mathbb{D}$. Set

$$F(z) = \frac{2}{\pi}\, i \,\mathrm{Log} \frac{1+z}{1-z}.$$

Show that $f \prec F$. (Log denotes the principal branch.)

(c) Let f be as in (b). Show that $|f'(0)| \leq \frac{4}{\pi}$.

5. Let (X, Σ, μ) be a measure space, Σ a σ-algebra of sets, μ a measure defined on Σ. Let $\mathcal{H} = L^2(X, \Sigma, \mu)$ be the Hilbert space of square-integrable Σ-measurable functions. Let $\Sigma_0 \subset \Sigma$ be a sub-σ-algebra of Σ and let $\mathcal{H}_0 = L^2(X, \Sigma_0, \mu)$ be the subspace of $\mathcal{H}$ consisting of functions in $\mathcal{H}$ which are Σ_0-measurable.

Let f be a function in $\mathcal{H}$. Show that there is a function f_0 in $\mathcal{H}_0$ (in particular Σ_0-measurable) such that

$$\int_X f_0(x) g(x)\, d\mu(x) = \int_X f(x) g(x)\, d\mu(x) \text{ for all } g \in \mathcal{H}_0.$$

Explain your reasoning.

6. (a) Let $F(x, y)$ be a continuous real-valued function defined on the closed square $[0,1] \times [0,1] \subset \mathbb{R}^2$. Show that

$$g(x) \equiv \sup_{y \in [0,1]} F(x, y)$$

is lower semi-continuous in the sense that for each $x \in [0,1]$ and $\epsilon > 0$, there is a δ such that

$$g(z) > g(x) - \epsilon \text{ for } |z - x| < \delta.$$

(b) Let $f(x)$ be a continuous real-valued function defined on $[0,1]$, and for $\kappa > 0$ define

$$A_\kappa \equiv \{x \in [0,1] : \ |f(x) - f(y)| \leq \kappa |x - y| \text{ for all } y \in [0,1]\}.$$

By considering $F(x, y) \equiv |f(x) - f(y)| - \kappa |x - y|$, show that the complement A_κ^c of A_κ is open, hence A_κ is closed.

(c) Show that for $\kappa > 0$,

$$B_\kappa \equiv \{x \in [0,1] : \ |f(y) - f(x)| < \kappa |x - y| \text{ for all } y \in [0,1], y \neq x\}$$

is a Borel measurable set of the real line.

7. Consider the real-valued function $F(x)$, $x \in \mathbb{R}$, defined by the (improper) Riemann integral

$$F(x) = \int_0^\infty \frac{\cos(xt)\, dt}{1+t}.$$

Show that $F(x)$ is a continuous function of x for $0 < x < \infty$. Hint: First integrate by parts to obtain an absolutely convergent integral.

8. Let $\{a_n\}_{n=1,2,\ldots}$ be a square summable sequence of complex numbers such that $\sum_n |a_n|^2 = 1$, and set

$$f_r(x) = \sum_{n \geq 1} r^n a_n e^{inx}$$

for $0 \leq r < 1$.

(a) Show that for each $r \in [0,1)$, the series defining $f_r(x)$ converges for each x, and that $f_r(x)$ is a bounded function of x. In particular, $f_r(x)$ is in $L^2([0,2\pi], dx)$, the space of square-integrable functions on the interval $[0, 2\pi]$ with Lebesgue measure.

(b) Let $\{r_j\}_{j=1,2,\ldots}$ be a sequence of numbers in $[0,1)$ such that $\lim_{j\to\infty} r_j = 1$. Show that the sequence of functions $\{f_{r_j}\}$ is an L^2-Cauchy sequence.

(c) Let

$$f = \lim_{j\to\infty} f_{r_j},$$

the limit meaning in an L^2-sense. Show that

$$a_m = \frac{1}{2\pi} \int_{[0,2\pi]} e^{-imx} f(x)\, dx$$

for all integers m.