Analysis General Exam, August 2009

Closed book, closed notes. Please pledge. In each problem, justify all assertions, show calculations, and identify those theorems that you invoke in your arguments.

1. Suppose that f is a real-valued function on the real line $\mathbf{R}$ with f absolutely continuous in the sense that it satisfies the fundamental theorem of calculus,

$$f(b) - f(a) = \int_a^b \frac{df(x)}{dx}\, dx$$

for $a, b \in \mathbf{R}$, and for df/dx an $L^1(\mathbf{R})$-function, i.e., integrable over $\mathbf{R}$.

a) Use the definition of the Lebesgue integral and basic theorems of real analysis to show that given $\epsilon > 0$, there exists a simple function g of compact support so that $\|g - df/dx\| < \epsilon$, with $\|\cdot\|$ the L^1 norm.

b) Prove that f is uniformly continuous and that $\lim_{x\to\infty} f(x)$ exists.

2. Let X be a measure space with measure μ and let $\{f_n\}$ be a sequence of real-valued measurable functions on X and f another measurable function so that

$$\mu\{x \in X : |f(x) - f_n(x)| \geq \frac{1}{2^n}\} < \frac{1}{2^n}.$$

a) Show that the sequence $\{f_n\}$ converges to f pointwise, a.e.

b) Suppose in addition that the functions of the above sequence are uniformly integrable in the sense that

$$\sup_n \int |f_n|\, d\mu \leq c$$

for some finite constant c. Show that f is integrable.

3. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane. For f analytic in $\mathbb{D}$ with power series

$$f(z) = \sum_{n\geq 0} a_n z^n,$$

let

$$\|f\|_2 = \left(\sum_{n\geq 0} |a_n|^2\right)^{1/2}.$$

Let $\mathcal{F}$ be the collection of such analytic functions with finite $\|\cdot\|_2$-norm.

a) Show that

$$\frac{1}{2\pi}\lim_{r\uparrow 1}\int_{\{z:|z|=r\}}|f(z)|^2\,|dz| = \|f\|_2^2$$

($|dz|$ is differential arclength). Hint: Use polar coordinates.

b) Show that if $\{f_k\}$ is a Cauchy sequence in $\mathcal{F}$ with respect to the norm $\|\cdot\|_2$ defined above, then $\{f_k(z)\}$ converges uniformly for $z \in \mathbf{K}$, $\mathbf{K}$ any compact set contained in $\mathbb{D}$, and that the limiting function is analytic in $\mathbb{D}$.

4. Suppose that $P(z)$ is a polynomial of degree $n \geq 2$ with n distinct zeros $z_1, z_2, \cdots, z_n$. Show that

$$\sum_{j=1}^{n}\frac{1}{P'(z_j)} = 0.$$

5. Suppose that f is analytic in a open set containing the upper half plane $\{z : \text{Im } z \geq 0\}$, and suppose further that for some $M, a > 0$, $|f(z)| \leq M/|z|^a$ for $|z|$ large. Show that for any z with Im $z > 0$,

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(x)}{x-z}dx.$$

6. True/False: Either prove that the statement is correct, or give a counterexample.

(a) If f is analytic in the entire complex plane $\mathbb{C}$ and Im$f \leq 0$ for all $z \in \mathbb{C}$, then f is constant.

(b) If f is analytic in the entire complex plane and bounded on the real axis, the f is constant.

(c) If f is analytic in the entire complex plane and $f(x+1) = f(x)$ for every real number x, then $f(z+1) = f(z)$ for every $z \in \mathbb{C}$.

(d) If f is analytic in the entire complex plane with $|f(z)| \geq |z|$ for all $z \in \mathbb{C}$ and $f(1) = 1$, then $f(z) = z$ for all $z \in \mathbb{C}$.

7. Suppose that f is an analytic map of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into itself that is not the identity function. Show that f can have at most one fixed point in $\mathbb{D}$.