Real Analysis, October 1977

1. Prove that the sequence

$$\sqrt{2}, \quad \sqrt{2\sqrt{2}}, \quad \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converges and find its limit.

2. For which real-x does

$$\lim_{n\to\infty} \frac{x^n}{1+x^{2n}}$$

converge? On which real sets is the convergence uniform?

3. (i) Give, if possible, an example of a function on $[0,\infty)$ whose improper Riemann integral exists and is finite, but which is not in $L^1(0,\infty)$ (Lebesgue measure).

   (ii) Give, if possible, an example of a sequence of Lebesgue integrable functions converging everywhere to a Lebesgue integrable function f, with

$$\lim_{n+\infty} \int_{-\infty}^{\infty} f_n(x)dx > \int_{-\infty}^{\infty} f(x)dx.$$

   (iii) Give, if possible, an example of a Riemann integrable function on $(0,1]$ whose points of discontinuity are dense.

4. For an interval $I = (a,\beta] \leq (0,1]$, let $X_I(x) = 1$ if $x \in I$ and 0 otherwise. For any simple function $f(x) = \Sigma a_j X_{I_j}(x)$, where $\{I_j = (a_j, \beta_j]\}$ is a finite disjoint collection of intervals define

$$\ell(f) = \Sigma a_j \left(\beta_j^{\frac{3}{4}} - a_j^{\frac{3}{4}}\right).$$

Prove

$$|\ell(f)|^2 \leq 2\int_0^1 |f(x)|^2 dx$$

for all f in $L^2(0,1)$.

5. State the Hahn and Jordan decomposition theorems for signed (real) measures. (Do not prove them.) Illustrate the theorems by considering the measure

$$\mu(\Delta) = \int_\Delta \sin t\, dt,$$

where Δ is a Borel subset of $[-1,1]$.

6. Prove that if a continuous linear functional ℓ on $C[-1,1]$ satisfies

$$\begin{aligned} \ell(1) &= 1, \\ \ell(f_n) &= 0, \text{ where } f_n(x) = x^n, \end{aligned}$$

$n = 1, 2, \ldots$, then

$$\ell(f) = f(0) \text{ for each } f \text{ in } C[-1,1].$$

HINT: Weierstrass approximation theorem.

7. Prove that

$$(*) \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t) \sin nt \, dt = 0,$$

if (i) $f(t) = X_{(a,b)}(t)$, where $-\infty < a < b < \infty$. Then deduce that (ii) $(*)$ holds if $f \in L^1(-\infty, \infty)$.

Real Analysis, April 1978

1. Show that the series
$$\sum_{k=1}^{\infty} \frac{\ell n(1+kx)}{kx^k}$$
converges uniformly on $[a,\infty)$ for every $a>1$.

2. If $r>1$, for what values of p, $1 \le p \le \infty$ does $x^{-1/r}(1+|\ell n\, x|)^{-1}$ belong to $L_p(0,\infty)$?

3. If λ and μ are σ-finite measures on $(X,\mathcal{B})$ with $\lambda << \mu$ and g denotes the Radon-Nikodym derivative of λ with respect to μ, prove that for every non-negative measurable function f,
$$\int_X f\, d\lambda = \int_X fg\, d\mu.$$

4. Let f be a real-valued function on $[0,1]$. Define the set C by
$$C = \{x \in [0,1] : f \text{ is continuous at } x\}.$$
Is C measurable? Justify your answer.

5. If $f \in L^1(0,1)$ and $\int_0^x f(t)dt = x$ for every $x \in (0,1)$ prove that $f(x) = 1$ a.e. on $(0,1)$.

6. Let M be a closed linear subspace of $L^1(0,1)$ and $f \in L^1(0,1)$ with $f \notin M$. Show that there exists a $g \in L^\infty(0,1)$ such that
$$\int_0^1 f(t)g(t)dt = 1,$$
and
$$\int_0^1 u(t)g(t)dt = 0, \qquad \text{all } u \in M.$$

7. Let $\{f_n\}$ be a sequence of measurable real-valued functions on the real $\mathbb{R}$ such that
$$\sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} |f_n(x)|dx < \infty.$$
Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for almost all $x \in \mathbb{R}$ and that for every Borel set E
$$\int_E \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \left(\int_E f_n(x)dx\right).$$

8. Let $(X,\mathcal{B},\mu)$ be a σ-finite measure space, $g \in L^1(X)$, and $\mathcal{A}$ a σ-algebra with $\mathcal{A} \subseteq \mathcal{B}$. Prove that there exists an $\mathcal{A}$-measurable function defined a.e. on X such that for every $A \in \mathcal{A}$.
$$(*) \qquad \int_A f\, d\mu = \int_A g\, d\mu.$$
Show that if f' is any other $\mathcal{A}$-measurable function satisfying $(*)$, then $f' = f$ a.e.

Real Analysis, September 1978

1. (a) State the Mean Value Theorem.

   (b) Let $f(x)$ be differentiable for $x > 0$. Prove that if

   $$\lim_{x\to\infty} f'(x) = 0,$$

   then

   $$\lim_{x\to\infty} [f(x+1) - f(x)] = 0.$$

2. Let $f_n(x)$ be a sequence of increasing functions on $0 \le x \le 1$ which converges pointwise to a function $f(x)$. Prove that if $f(x)$ is continuous, then $f_n(x)$ converges *uniformly* to $f(x)$.

3. For $\alpha > 0$ and $\beta > 0$, let

   $$f(x) \begin{cases} x^\alpha \sin\left(\frac{1}{x^\beta}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

   (a) For what values of α and β is $f(x)$ continuous?

   (b) For what values of α and β is $f(x)$ of bounded variation on $-1 \le x \le 1$?

4. Assume that $f(x) \in L_p[0,\infty)$ for some p, $1 \le p < \infty$, and that $f(x)$ is uniformly continuous on $[0,\infty)$. Prove that

   $$\lim_{x\to\infty} f(x) = 0.$$

5. Let $K(x,y)$ be a measurable function on the square $0 \le x$, $y \le 1$ satisfying

   $$\int_0^1 \int_0^1 |K(x,y)|^2 dx\, dy < \infty.$$

   Prove that if $f \in L_2[0,1]$, then the integral

   $$g(x) = \int_0^1 K(x,y) f(y) dy$$

   converges for a.e. x and that $g \in L_2[0,1]$.

6. Let μ and ν be finite Borel measures on $[0,\infty]$.

   (a) Prove that there is a unique finite Borel measure on $[0,\infty)$ such that for every bounded continuous function $\phi(x)$,

   $$\int_0^\infty \phi(x) d\lambda(x) = \int_0^\infty \int_0^\infty \phi(xy) d\mu(x) d\nu(x).$$

(b) If $d\mu(x) = h(x)dx$ is absolutely continuous (with respect to Lebesgue measure), prove that λ is also absolutely continuous, and compute its Radon-Nikodym derivative.

7. State Egorov's Theorem. Give a counter-example on an infinite measure space.

8. (a) Suppose $f,\, g \in L_1(-\infty, \infty)$. Prove that

$$h(x) = \int_{-\infty}^{\infty} f(x+t)g(t)dt$$

exists for almost all real x and defines a function in $L_1(-\infty, \infty)$.

(b) Suppose $g \in L_1(-1, 1)$ and

$$\int_{-1}^{1} x^n g(x)dx = 0 \text{ for } n = 0, 1, \ldots$$

Prove $g = 0$ a.e.

Real Analysis, January 1979

1. Find the maximum of the expression

$$\frac{x - \frac{y}{2} + \frac{z}{3}}{x^4 + y^4 + z^4}$$

over the region $x^4 + y^4 + z^4 \geq 1$.

2. Let $f(x)$ be a function of bounded variation on $(-\infty, \infty)$. Suppose that $F(x)$ has compact support and that each continuously differential function $f(x)$ satisfies

$$\int_{-\infty}^{\infty} F(x) f'(x) dx = 0.$$

Show that $F(x) = 0$.

3. Suppose that $\{f_n(x)\}$ is a sequence of measurable functions on $[0, 1]$ for which

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ exists a.e.}$$

Suppose that

$$\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)| dx = 0,$$

and that

$$\sup_n \int_0^1 |f_n(x)|^2 dx < \infty.$$

Show that $\int_0^1 |f(x)|^2 dx < \infty$. Does it follow that $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2 dx$. (Prove or give a counter-example.)

4. Let $f(x)$ be measurable on $(-\infty, +\infty)$ and satisfy

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

Show that for each h the integral

$$\int_{-\infty}^{\infty} f(x) f(x+h) dx$$

converges absolutely. Define

$$g(h) \equiv \int_{-\infty}^{\infty} f(x) f(x+h) dx,$$

and show that $g(h)$ is continuous and that

$$\lim_{|h|\to\infty} g(h) = 0.$$

5. Let $f(\lambda, x)$ be a continuous function of two variables, $0 < \lambda < 1$ and $0 < x < 1$. Suppose that $\frac{\partial f}{\partial \lambda}(\lambda, x)$ exists for all λ and x and that

$$h(x) = \sup_{0<\lambda<1} \left| \frac{\partial f}{\partial \lambda}(\lambda, x) \right|$$

satisfies $\int_0^1 h(x)dx < \infty$.

Show that the function $F(\lambda) \equiv \int_0^1 f(\lambda, x)dx$ is differentiable and satisfies

$$F'(\lambda) = \int_0^1 \frac{\partial f}{\partial \lambda}(\lambda, x)dx.$$

6. Show that a convex function defined on an open interval is continuous.

7. Does there exist a nowhere dense set of positive Lebesgue measure in $[0, 1]$? Prove or give an example.

Real Analysis, January 1980

1. Let $f(x)$ be a continuous function on $[0,1]$ such that
$$\int_0^1 x^n f(x)dx = 0; \qquad \text{for } n = 0,1,2,\ldots$$
Either prove that $f(x) \equiv 0$ or give a counter-example.

2. Let $f(x)$ be an integrable function on $[0,1]$. Either prove or give a counter-example that
$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0.$$

3. Let $\sum_{n=0}^{\infty} a_n e^{-nx}$ be a series which converges at $x = 0$. Prove that the series converges uniformly on the interval $[1,\infty)$.

4. Let $\{f_n(x)\}$ be a sequence of continuous functions on $[0,\infty)$. Show that the set E of points x for which $\{f_n(x)\}$ converges is Borel measurable.

5. Let $\{a_k\}$ be a sequence such that the series $\sum_{k=1}^{\infty} a_k x_k$ converges whenever $\sum_{k=1}^{\infty} |x_k|^2 < \infty$. Show that $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

6. Prove that if $a_n \geq 0$ $(n = 0,1,2,\ldots)$ and if
$$f(t) = \sum_{n=0}^{\infty} a_n t^n \leq c \quad (0 \leq t < 1),$$
then the function $f(t)$ has a limit as $t \to 1$, equal to
$$\sum_{n=0}^{\infty} a_n.$$

7. Let $\mu(dt)$ and $\nu(dt)$ be two finite Borel measures on the half line $(0,\infty)$. Show that there is a unique finite measure $\omega(dx)$ and $(0,\infty)$ which satisfies
$$\int_0^{\infty} f(z)\omega(dx) = \int_0^{\infty} \left\{ \int_0^{\infty} f(st)\mu(dt) \right\} \nu(ds)$$
for each bounded continuous function f on $(0,\infty)$.

   Find an explicit formula for $\omega(dx)$.

Real Analysis, September 1980

1. Define
$$g(x) = \begin{cases} |x|^\alpha \log|x| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Find all α such that $g'(0)$ exists.

2. True or False. (If true, give a proof; if false, give a counter-example.) If f is a real-valued differentiable function on $(0,1)$, then f is measurable on $(0,1)$.

3. Let $f(x,t)$ be a real-valued function on $\mathbb{R}^2$ such that for all $t \in \mathbb{R}$, $f(\quad, t)$ is continuous. Suppose there exists a Lebesgue integrable function $g(t)$ such that
$$|f(x,t)| \leq g(t), \quad \text{for all } x, t \in \mathbb{R}.$$
Prove that the function
$$F(x) = \int_{\mathbb{R}} f(x,t) \cos t \, dt$$
is finite and continuous.

4. Prove that if $\{a_{n,m} : n = 1, 2, \ldots; \; m = 1, 2, \ldots\}$ are non-negative real numbers, then
$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m}.$$

5. Prove that $C[0,1]$, the continuous complex functions on $[0,1]$, with inner product $(f,g) = \int_0^1 fg \, dx$ is not a Hilbert space.

6. Let $\alpha(x)$ be a monotone increasing function on $[0,1]$ such that $\alpha'(x)$ exists a.e. on $[0,1]$. Prove directly that
$$\alpha(1) - \alpha(0) \geq \int_0^1 \alpha'(x) dx.$$
Give an example to show equality does not always hold.

7. (a) Let h be a non-negative Lebesgue integrable function on $\mathbb{R}$. For each measurable set $E \subset \mathbb{R}$, define
$$\mu(E) = \int_E h(x) d\lambda(x),$$
where λ is Lebesgue measure.
Prove directly that if f is a real-valued function on $\mathbb{R}$, then f is integrable with respect to μ if and only if fh is integrable with respect to λ and if f is integrable
$$\int_{\mathbb{R}} f \, d\mu = \int_{\mathbb{R}} fh \, d\lambda.$$
(b) Find the Radon-Nikodym derivative of μ with respect to λ, i.e., $\frac{d\mu}{d\lambda}$. Defend your selection.

Real Analysis, January 1981

1. **Definition.** A real-valued function defined on an interval of $\mathbb{R}^1$ has a relative maximum at x if there exists a neighborhood N of x such that $f(y) \le f(x)$ for all $y \in N$. It has a proper maximum at x if there exists a neighborhood N of x such that $f(y) < f(x)$ for all $y \in N$.

   Prove that any real-valued function defined on an interval of $\mathbb{R}^1$ has only a countable number of proper maximums.

   Give an example to show that the statement is false if proper maximum is replaced by relative maximum.

2. Let a and c be real numbers with $c > 0$. Define f on $[0,1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-c}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   Prove:

   (a) f is continuous iff $a > 0$;

   (b) $f'(0)$ exists iff $a > 1$;

   (c) f' is bounded iff $a \ge 1 + c$.

3. Prove or give a counter-example: If f is differentiable on $(0,1)$, then its derivative is a measurable function on $(0,1)$.

4. (a) Show that $\frac{\sin x}{(1+x)^2}$ is Lebesgue integrable on $(0,\infty)$ but $\frac{\cos x}{1+x}$ is not.

   (b) Prove that

$$\lim_{A\to\infty} \int_0^A \frac{\cos x}{1+x}\, dx = \int_0^\infty \frac{\cos x}{1+x}\, dx$$

   exists, and

$$\int_0^\infty \frac{\cos x}{1+x}\, dx = \int_{(0,\infty)} \frac{\sin x}{(1+x)^2}\, dx.$$

5. Let $f(x,t)$ be a function defined on $(0,1)\times(0,1)$ to $\mathbb{R}^1$ such that

   (a) $\frac{\partial}{\partial t} f(\quad, t_0)$ exists for all $t_0 \in (0,1)$;

   (b) there exists $g \in L^1(0,1)$ such that

$$|f(x,t_1) - f(x,t_2)| \le g(x)|t_1 - t_2|$$

   for all $x, t_1, t_2 \in (0,1)$.

   Prove that $f(\,\cdot\,,t) \in L^1(0,1)$ for each $t \in (0,1)$, that $\frac{\partial f}{\partial t}(\,\cdot\,,t) \in L^1(0,1)$ for each $t \in (0,1)$ and that

$$\frac{d}{dt}\int_0^1 f(x,t)dx = \int_0^1 \frac{\partial f}{\partial t}(x,t)dx.$$

6. Assume as known that every set in $\mathbb{R}^n$ of positive Lebesgue measure contains a non-measurable set.

Let T be a bijection of a bounded measurable set $E \subset \mathbb{R}^n$ to $T(E) \subset \mathbb{R}^n$ such that $T(A)$ is measurable if and only if A is measurable. Let μ be Lebesgue measure on E.

(a) Prove that T is a measurable map, i.e., $T^{-1}C$ is measurable for each measurable $C \subset T(E)$.

(b) Prove that the measure μT defined on measurable subsets of E by $\mu T(A) = \mu(TA)$ is absolutely continuous with respect to μ.

(c) Prove that μT is equivalent to μ.

Real Analysis, September 1981

1. Let $f_n(x) = \frac{x}{1+nx^2}$, $-\infty < x < \infty$.
   (a) Find $f(x) = \lim f_n(x)$ and $g(x) = \lim f_n'(x)$.
   (b) Where is the convergence uniform?
   (c) Is f differentiable? If $f'(x) = g(x)$ for all x?

2. Let

$$\begin{aligned} f(x) &= x^a \sin\left(\frac{1}{x}\right), \quad 0 < x \le 1; \\ f(0) &= 0. \end{aligned}$$

   For what values of $a > 0$ is f of bounded variation on $[0,1]$.

3. Let $f(x)$ be a real-valued function on R, with

$$|f(x) - f(y)| \le (x-y)^2.$$

   For all x and y. What can be said about $f(x)$?

4. Let μ be a measure on the space X. Let f be a non-negative μ-integrable function from X to $\mathbb{R}$. Show that $\{x \in X : f(x) > 0\}$ can be written as a countable union of sets each having finite μ measure.

5. Let $f \in L^1[0,1]$. Let $S = \{x \in [0,1] : f(x) \text{ is an integer}\}$. Evaluate

$$\lim_{n\to\infty} \int_0^1 |\cos \pi f(x)|^n dx.$$

6. Let f and g be Lebesgue measurable functions on $\mathbb{R}^n$, and suppose
   (a) f, g are both in $L^1(\mathbb{R}^n)$,
   (b) f, g are both in $L^2(\mathbb{R}^n)$,
   (c) f is in $L^1(\mathbb{R}^n)$, g is in $L^\infty(\mathbb{R}^n)$.

   Which of the hypotheses, (a), (b), or (c), imply that the product fg is in $L^1(\mathbb{R}^n)$?

7. Suppose E is a Lebesgue measurable subset of $\mathbb{R}^n$ such that $\mu(E) = 1$. Prove that E contains a Lebesgue measurable subset A such that $\mu(A) = \frac{1}{2}$. Give reasons.

Real Analysis, January 1982

True or False. First tell whether the statement is a true statement or a false statement. If you claim that it is true, give a proof of the statement. If you claim that it is false, give a counter-example.

1. Let $\{f_n\}$ is a bounded sequence of integrable functions (i.e., with finite integrals) on a measure space $(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ a.e., then $\int_\Omega f_n d\mu \to \int_\Omega f\, d\mu$.

2. Let λ denote Lebesgue measure in R^2. If
$$B = \{(x, y) \in R^2 :\ 0 \le x \le 1,\ x \le y \le x^3 + 1\},$$
then B is measurable and $\lambda(B) = \frac{3}{4}$.

3. If μ is a regular Borel measure on $[0, 1]$, then $\mu\{x\} = 0$ for all $x \in [0, 1]$.

4. If $(\Omega, \mathcal{F}, \mu)$ is a probability space and $\{f_n\}$ is a sequence of functions in $L^p(\Omega, \mathcal{F}, \mu)$, when $1 \le p \le \infty$, and $f_n \xrightarrow{L^p} f$, then $f_n \xrightarrow{L^r} f$ for all $r > p$.

5. If $\{f_n\}$ is a sequence of measurable functions on $(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ a.e., then $f_n \to f$ in measure. What if $\mu(\Omega) < \infty$?

Real Analysis, September 1982

1. Is the sequence $f_n(x) = \frac{1}{1+nx}$ uniformly convergent on the inteval $[0, 1]$? Explain why or why not.

2. For what values of $p > 0$ is the function

$$f(x) = x^p \sin \frac{1}{x}$$

   of bounded variation on $[0, 1]$?

3. Assume that $a_n > 0$ and that $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that

$$\sum_{n=1}^{\infty} \sqrt{a_{n+1}a_n}$$

   is convergent.

4. Let $r_1, r_2, \ldots$ be an enumeration of the rationals, and let

$$I_k = \left(r_k - \frac{1}{k^2}, r_k + \frac{1}{k^2} \right) \quad (k = 1, 2, \ldots).$$

   Is there a point x which is not in $\bigcup_{k=1}^{\infty} I_k$? Explain.

5. Let E be a subset of real line Lebesgue measure zero. Can the set

$$G = \{(x, y) : x - y \in E\}$$

   have positive two-dimensional Lebesgue measure? (Hint: Change variables and use Fubini's theorem.)

6. A function on $[0, \infty)$ is *improperly Riemann-integrable* iff the Riemann integral

$$I(t) = \int_0^t f(x)dx \quad \text{exists for every } t > 0,$$

   and if

$$I(t) = \lim_{t \to \infty} \int_0^t f(x)dx \quad \text{exists.}$$

   Prove that if $f(x)$ is continuous on $[0, \infty)$ and $|f(x)|$ is improperly Riemann integrable, then

   (a) $f(x)$ is improperly Riemann integrable,

   (b) $f(x)$ is Lebesgue integrable on $[0, \infty]$, and

   (c) $I = \int_0^{\infty} f \, dm$,

   where m is Lebesgue measurable.

Real Analysis, January 1983

1. Let $f(x)$ be continuous on $[-1,1]$ and differentiable except possibly at $x=0$. Prove that if
$$\lim_{x\to 0} f'(x) = A$$
exists, then $f'(0)$ exists and is equal to A.

2. Let $f_n(x)$ be a sequence of differentiable functions, whose derivatives $f_n'(x)$ are continuous on $[0,1]$. Prove that if the sequence of derivatives $f_n'(x)$ converges uniformly on $[0,1]$ to a function $g(x)$, and $f_n(0)$ converges, then $f_n(x)$ converges to a continuously differentiable function $f(x)$ with $f'(x) = g(x)$.

3. Prove or give a counter-example:
   (a) If $\{f_n\}$ is a decreasing sequence of measurable non-negative functions on $[0,1]$, and $f_n \to 0$ a.e., then $\int_0^1 f_n(x)dx \to 0$.
   (b) A bounded measureable function on $[0,1]$ can be made continuous by redefining it on a set of measure zero.
   (c) If $f_n \to f$ in measure on $[0,1]$, then $f_n \to f$ a.e.
   (d) If $a_{ij} \geq 0$ for all i, j, then
$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right).$$

4. Let f be a real-valued Lebesgue measurable function on $[0,1]$.
   (a) Define the essential supremum of f, ess sup f.
   (b) Prove that there exists a real-valued Lebesgue measurable function g on $[0,1]$ such that
$$\sup g = \text{ess sup } f,$$
and $f(x) = g(x)$ for almost all $x \in [0,1]$.

5. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by
$$F(x) = \begin{cases} 0, & x < 1, \\ (x-1)^2 + 1, & 1 \leq x < 2, \\ 2, & 2 \leq x. \end{cases}$$
Let μ be the Borel measure on $\mathbb{R}$ such that
$$\mu((-\infty, x]) = F(x) \quad \text{for } x \in \mathbb{R}.$$
Find the Lebesgue decomposition
$$\mu = \mu_a + \mu_s$$
of μ, where μ_a is absolutely continuous with respect to Lebesgue measure, and μ_s is singular. Give reasons for your answer.

6. Let μ be a Borel measure on $[0,1]$. Show that

$$\int_0^1 |s-t|^{-\frac{1}{2}} d\mu(t)$$

exists and is finite for almost all s in $\mathbb{R}$ (with respect to Lebesgue measure). Hint: Estimate, for each $A > 0$,

$$\int_{[-A,A]} \int_{\mathbb{R}} |s-t|^{-\frac{1}{2}} d\mu(t) ds.$$

Real Analysis, September 1983

1. Give an example of a sequence of measurable functions on $[0,1]$ which

   (a) converges to zero a.e., but not in $L^1[0,1]$,

   (b) converges to zero in $L^1[0,1]$, but
   $$\lim f_n(x)$$
   fails to exist for a.e. x.

2. Let f be a non-negative Lebesgue measurable function on $[0,1]$.
   $$S(f) = \{(x,y)|0 < y < f(x)\} \text{ and } G(f) = \{(x,y)|y = f(x)\}.$$
   Prove that $S(f)$ and $G(f)$ are Lebesgue measurable in $\mathbb{R}^2$, and find their measures.

3. A subset E of $\mathbb{R}$ is *locally measurable* if every point x of R has a neighborhood of $(x-\delta, x+\delta)$ such that $E \cap (x-\delta, x+\delta)$ is measurable. Prove that every locally measurable set is measurable.

4. Do the following sequences converge uniformly on $[0,1]$?

   (a) $f_n(x) = nx(1-x)^n$;

   (b) $g_n(x) = \frac{x}{1+nx}$.

5. Let $f(\theta)$ be 2π-periodic and twice continuously differentiable. Prove that its Fourier series converges uniformly. You do not need to prove the series converges to f. **OR** Prove the Riemann-Lebesgue lemma for $L^1[0,2\pi]$-functions
   $$\lim_{n\to\infty} \int_0^{2\pi} e^{inx} f(x)dx = 0.$$

Real Analysis, September 1984

1. Let f and g be Lebesgue measurable functions on $\mathbb{R}^n$. Consider the hypotheses:

   (i) f, g are both in $L^1(\mathbb{R}^n)$,

   (ii) f, g are both in $L^2(\mathbb{R}^n)$,

   (iii) f is in $L^1(\mathbb{R}^n)$, g is in $L^\infty(\mathbb{R}^n)$.

   (a) Which of the hypotheses imply fg is in $L^1(\mathbb{R}^n)$?

   (b) For those hypotheses which do not imply fg is in $L^1(\mathbb{R}^n)$ give a counter-example.

2. Suppose $\{x_i\}$ is any sequence of points on $\mathbb{R}$ and let $p_i \geq 0$ with $\sum_i^\infty p_i = 1$. Define

$$F(x) = \sum_{\substack{i \\ x_i \leq x}} p_i,$$

   and let μ_F denote Lebesgue-Stieltjes measure with respect to F. Show that

   (i) Every function $f : \mathbb{R} \to \mathbb{R}$ is μ_F-measurable,

   (ii) f is integrable if and only if $\sum_i p_i f(x_i)$ converges absolutely.

3. Prove one of the following:

   (i) The rational numbers Q are a G_δ in $\mathbb{R}$.

   (ii) The rational numvers Q are not a G_δ in $\mathbb{R}$.

4. State and prove the Riemann-Lebesgue lemma for functions on the circle.

5. Let E and F be sets of positive Lebesgue measure on $\mathbb{R}$. Prove that some translate of F, i.e., $F_a = \{x | (x-a) \in F\}$ intersects E in a set of positive measure.

6. (i) Define the Riemann integral.

   (ii) Let, for $0 \leq x \leq 1$,

$$f(x) = \begin{cases} \dfrac{1}{2^n} & \text{if } x = \dfrac{p}{2^n},\ p \text{ odd}, \\ 0, & \text{otherwise.} \end{cases}$$

   Using the definition given in (i), decide whether f is Riemann integrable. Prove your answer.

7. (i) Suppose $f_n \in L^1(\mathbb{R})$ is a fast Cauchy sequence in the sense that $\|f_n - f_{n-1}\| < \frac{1}{2^n}$, where $\|\cdot\|$ denotes the L^1-norm. Prove that

$$\lim_{n\to\infty} f_n(x)$$

   exists a.e.

   (ii) Give an example of an L^1-Cauchy sequence $\{f_n\}$ such that f_n does not converge pointwise a.e.

Real Analysis, September 1985

1. Evaluate
$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\left(1-e^{-t^2/n}\right)e^{-|t|}\sin^2 t\,dt.$$
Justify each step in your computation.

2. (a) Let $\{u_n(x)\}$, $n=0,1,2,\ldots$ be a sequence of functions defined iteratively by
$$u_n(x)=u_{n-1}(x)\,(1-u_{n-1}(x))\,,$$
with $u_0(x)=g(x)$ a continuous function $0\le g(x)\le 1$. Prove that $\{u_n(x)\}$ converges uniformly, $n\to\infty$.

   (b) Let $\{r_n\}$ be an enumeration of the rationals and define the step function
$$\begin{aligned} j_n(x) &= 0, \quad x<r_n,\\ &= 1, \quad x\ge r_n. \end{aligned}$$
Let
$$f(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}\,j_n(x).$$
Prove that $f(x)$ is continuous at any irrational point x_0.

3. (a) Assuming the monotone convergence theorem, give a proof of Fatou's lemma: For any sequence of Lebesgue integrable $f_n\ge 0$,
$$\int \liminf f_n(x)dx \le \liminf \int f_n(x)dx.$$

   (b) Give an example where the inequality is strict.

4. (a) Is $\sum_{n\ge 0}\sin(n)e^{inx}$ the Fourier series for an $L^1([0,2\pi],dx)$ function? Explain your answer briefly.

   (b) Is $\sum_{n\ge 1}^{\infty} n^{-3/4}e^{inx}$ the Fourier series for an $L^2([0,2\pi],dx)$ function? Explain your answer briefly.

5. (a) For what real values of α is
$$\Lambda_\alpha(f)\equiv\int_0^1 x^\alpha f(x)dx$$
a continuous linear functional on $L^2([0,1],dx)$?

   (b) Set
$$\|\Lambda_\alpha\|=\sup\left\{\left|\int_0^1 x^\alpha f(x)dx\right|\,\middle|\,\int_0^1|f(x)|^2dx=1\right\}.$$
Compute $\|\Lambda_\alpha\|$.

6. Let $f\in L^2(\mathbb{R},dx)$, and let $\hat f$ be its Fourier transform. Suppose that $\hat f(k)(1+|k|)\in L^2(\mathbb{R},dk)$. Prove that f is equal, a.e., to a continuous function. (HINT: Show $\hat f$ is an L^1-function.)

Real Analysis, January 1986

1. (a) Let $g(x)$ be defined
$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{inx}.$$
Is $g(x)$ continuous? Why?

   (b) Let f be an $L^1(\mathbb{R})$-function. Prove that
$$g(x) = \int e^{ixy} f(y) dy$$
is a continuous function.

2. Prove the Riemann-Lebesgue lemma: If $f \in L^1(\mathbb{R})$,
$$\lim_{x\to\infty} \int e^{ixy} f(y) dy = 0.$$

3. Which of the following functions are Lebesgue integrable over $\mathbb{R}$?

   (i) $f(x) = \dfrac{\sin x}{x}, \quad x \neq 0,$

   (ii) $f(x) = e^{-|x|}|x|^{\frac{1}{2}} \cos x^2.$

   Prove your conclusions. For which $p > 1$ are these functions in L^p?

4. For intervals $I \subset \mathbb{R}$ define $\mu(I)$ to be the length of I. Without using any results of general measure theory, show that μ is a σ-additive set of function on the class of intervals. Then state a theorem which extends μ to a measure on a suitable σ-field in $\mathbb{R}$.

5. Prove
$$\lim_{\alpha\downarrow 0} \int \frac{e^{\frac{-|x|}{\alpha}}}{2\alpha} f(x) dx = f(0)$$
for any continuous function f of compact support. To do this, show that for any $\alpha > 0$,
$$\int \frac{e^{\frac{-|x|}{\alpha}}}{2\alpha} dx = 1,$$
and that for any $\delta > 0$,
$$\int_{|x|>\delta} \frac{e^{\frac{-|x|}{\alpha}}}{2\alpha} dx \to 0$$
for $\alpha \to 0$.

6. Give examples of two functions $F(x)$, $G(x)$ defined on $\mathbb{R}$ such that

   (i) $x_1 > x_2 \Rightarrow F(x_1) > F(x_2)$ and $G(x_1) > G(x_2)$;

(ii) $F'(x) = G'(x) = 0$ a.e.;

(iii) $F(x)$ has discontinuities on a countable set $\{x_i\}$ and $\sum(F(x_i+0) - F(x_i-0)) = 1$;

(iv) $G(x)$ is continuous everywhere.

7. Suppose that $\{f_n\}$ is a Cauchy sequence of real-valued L^2($\mathbb{R}$-functions (in the sense that for any $\epsilon > 0 \exists N$ such that $\|f_n - f_m\|_2 < \epsilon$ if $n, m > N$).

   (a) Give an example of such a sequence which does not converge pointwise.

   (b) Let Λ be the linear functional on $L^2(\mathbb{R})$ defined by

$$\Lambda(f) = \int \frac{1}{1+|x|} f(x)dx.$$

   Prove that the real-valued sequence $\{\Lambda(f_n)\}$ is Cauchy if $\{f_n\}$ is an L^2-Cauchy sequence.

8. For real y, let $((y))$ denote the unique real in $[0, 1)$ such that, for some $n \in \mathbb{Z}$, $n + ((y)) = y$. For fixed $x \in \mathbb{R}$, define

$$u_k(x) = ((kx)), \quad k = 1, 2, \ldots$$

Real Analysis, September 1986

1. Evaluate, and explain briefly your reasoning:

$$\lim_{n\to\infty}\int_0^1 \frac{\sin(nx)}{\sqrt{x}}\,dx.$$

2. Let

$$f(x)=\sum_{n=-\infty}^{+\infty}\frac{e^{inx}}{2^{|n|}}.$$

   (a) Is $f(x)$ continuous? Differentiable? Infinitely differentiable? Explain briefly your reasoning.

   (b) Evaluate, and explain briefly your reasoning:

$$\int_0^{2\pi}|f(x)|^2dx.$$

3. (a) Let $F(x)$ be a *continuous* nondecreasing function on $[0,1]$. When is

$$F(x)=\int_0^x F'(t)dt?$$

   (b) Let

$$F(x)=\sum_{\substack{1/n\le x\\ n\ge 2}}3^{-n}=\sum_{n=2}^{\infty}3^{-n}H(x-1/n),$$

   where $H(x)=0,\ x<0,\ =1,\ x\ge 0$.
   Express $\int_0^1 g(x)dF(x)$ as an infinite series, where $g(x)$ is continuous on $[0,1]$.

4. Let $\pi : Z^+\to Z^+$ be a permutation of the positive integers $Z^+=\{1,2,\ldots\}$. Prove that if $a_n\ge 0$,

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}a_{\pi(n)},$$

   i.e., any rearrangement of a series of positive terms converges to the same result.

5. Prove, by showing directly the convergence of the difference quotient, that

$$\frac{d}{dy}\int_0^1\frac{\sin(xy)}{\sqrt{x}}\,dx=\int_0^1\cos(xy)\,\sqrt{x}\,dx.$$

6. Prove: If $f_n\to f$ is $L_1[0,1]$ and $f_n(x)\to g(x)$ a.e., then $f(x)=g(x)$ a.e.

7. Prove that the uniform limit of a sequence of real-valued continuous functions $\{f_n\}$ on $[0,1]$ is continuous.

Real Analysis, January 1987

1. Give an example of a sequence $f_n \geq 0$ of functions on $\mathbb{R}$ satisfying

   (i) $\lim f_n(x) = 0$ a.e., but $\int f_n$ does not tend to zero.

   (ii) $\int f_n \to 0$, but $f_n(x)$ does not tend to zero a.e.

2. Let $\sum_{n=1}^{\infty} |a_n| < \infty$. Prove that

$$\lim_{\epsilon \downarrow 0} \sum_{n=1}^{\infty} e^{-\epsilon n} a_n = \sum_{n=1}^{\infty} a_n.$$

3. Let $\mathcal{F}$ be a σ-field on X, and $f(x)$ a real-valued function on X.

   (a) Define what it means for f to be $\mathcal{F}$-measurable.

   (b) Let $f(x) = x$ on $\mathbb{R}$. What is the smallest σ-field $\mathcal{F}$ of subsets for $\mathbb{R}$ for which f is $\mathcal{F}$-measurable?

4. (a) Show that $\sum_{n=1}^{\infty} \frac{e^{inx}}{n^{1+\alpha}}$ converges to a continuous function $\alpha > 0$.

   (b) Show that $\sum_{n=1}^{\infty} \frac{e^{inx}}{n^{\alpha}}$ is the Fourier series of a function in $L^2[0, 2\pi]$ if $\alpha > \frac{1}{2}$.

5. Let $F(x)$ be a monotone increasing continuous function with $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. Regarding F as the distribution of a Stieltjes measure μ, compute, in terms of F, the measure of

   (a) $[0, 1]$,

   (b) the set consisting of all rationals, and

   (c) the set $(x : \; F(x) > b)$.

6. Prove that

$$\lim_{n\to\infty} n \int_0^{\infty} e^{-x^2}(e^{x/n} - 1)dx = \int_0^{\infty} xe^{-x^2} dx = \frac{1}{2},$$

   and justify your procedure.

7. (a) Suppose that f and g are $L^1(\mathbb{R})$ functions. Show that

$$F(x) = \int f(x-y)g(y)dy$$

   defines an L^1 function.

   (b) Suppose instead that f and g are $L^2(\mathbb{R})$ functions. Show that $F(x)$ is a bounded function.

   (c) Show further, for f and g in $L^2(\mathbb{R})$, that $\lim_{x\to\pm\infty} F(x) = 0$.

Real Analysis, September 1987

1. Let $\{f_n\}_1^\infty$ and f be functions in $L^1(0,1)$. Determine which of the following assertions are true. If true, give a proof, and if false, give a counter-example.

   (a) If $\lim_{n\to\infty} \int_0^1 |f(x) - f_n(x)|dx = 0$, then $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.

   (b) If $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., then $\lim_{n\to\infty} \int_0^1 |f(x) - f_n(x)|dx = 0$.

   (c) If $\lim_{n\to\infty} \int_0^1 |f(x) - f_n(x)|^2 dx = 0$, then $\lim_{n\to\infty} \int_0^1 |f(x) - f_n(x)|dx = 0$.

   (d) If $\lim_{n\to\infty} \int_0^1 |f(x) - f_n(x)|dx = 0$, then $\lim_{n\to\infty} \int_0^1 |f(x) - f_n(x)|^2 dx = 0$.

2. Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2}, \quad x \in \mathbb{R}^1.$$

   (a) For what values of x does the series converge absolutely?

   (b) For $\epsilon > 0$, does the series converge uniformly on $[\epsilon, \infty)$?

   (c) Is f bounded on the set where it is finite?

3. Let $f(x) = 1_{[0,\infty)}(x)$. Let $\{x_n\}$ be any countable set of real numbers, and let λ_n be any sequence of positive numbers with $M = \sum_{n=1}^{\infty} \lambda_n < \infty$. Define the function

$$F(x) = \sum_{n=1}^{\infty} \lambda_n f(x - x_n).$$

   (a) Show that $F(x)$ is bounded, monotonic, non-decreasing, and right continuous.

   (b) Show that all discontinuities of $F(x)$ are contained in the set $\{x_n\}$.

   (c) Is it true that for a.e. x, $F'(x)$ exists and $F'(x) = 0$? Justify your answer.

4. True or false with proof or counter-example.

   Let $A \subseteq \mathbb{R}^1$ be an open set which contains all rational numbers. Then $A = \mathbb{R}^1$.

5. Suppose f and g are *non-negative* functions in $L^1(\mathbb{R}^1, dx)$, and let

$$f * g(x) \equiv \int_{-\infty}^{\infty} f(x - y)g(y)dy.$$

   (a) Show that $f^* g(x) < \infty$ for a.a. x.

   (b) Show that

$$\int_{-\infty}^{\infty} f * g(x)dx = \left(\int_{-\infty}^{\infty} f(x)dx\right)\left(\int_{-\infty}^{\infty} g(x)dx\right).$$

6. Suppose $f(x)$ is a real-valued twice continuously differentiable periodic function with period 2π.

   (a) Show that the Fourier series of $f(x)$ converges for each x.

   (b) Show that for each x the Fourier series of f converges to $f(x)$.

Real Analysis, May 1988

Two-part exam. Part I counts as Statistics Real Variable Exam.
ALL L^p-SPACES ARE WITH RESPECT TO LEBESGUE MEASURE dx. ALL FUNCTIONS ARE FINITE-VALUED.

PART I

1. Let $f_1, f_2, \ldots$ be a seuqnce of non-negative, continuous functions on the interval $[0, 1]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for each x in the interval.
   (a) Is $f(x)$ necessarily continuous?
   (b) Is $f(x)$ necessarily Lebesgue measurable?
   (c) Is $f(x)$ necessarily Lebesgue integrable?
   (d) Is $\int_0^1 f(x)dx = 1$ if each $\int_0^1 f_n(x)dx = 1$?

   In each case justify your answer by invoking a theorem of giving a counter-example.

2. Let
$$f(x) = \sum_{n=0}^{\infty} 2^{-n} e^{inx}.$$
   (a) Is f continuous?
   (b) Is f differentiable?
   (c) Evaluate $\int_0^{2\pi} |f(x)|^2 dx$.

   In each case, explain your reasoning.

3. For $c > 1$, find
$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-cx} dx,$$
   and justify your answer in detail.

4. If f is continuous on the interval $[0, 2\pi]$ and differentiable there except perhaps at $x = 1$, but $\lim_{x\to 1} f'(x) = c$ exists, show that $f'(1) = c$.

5. Let E be a measurable subset of the unit square in the plane with the property that for each x in $[0, 1]$ the x-slice $E_x = \{y \in \mathbb{R} | (x, y) \in E\}$ is a finite set. Prove that E has planar Lebesgue measure zero.

PART II

6. For a fixed function f in $L^1(\mathbb{R})$ (with respect to Lebesgue measure dx) define the translates of f by
$$f_t(x) = f(x+t).$$
Show in detail that the map $t \to f_t$ is a continuous map of $\mathbb{R}$ into the metric space $L^1(\mathbb{R})$.

7. Consider the sequence of functions $f_n(x) = \cos(nx)$ on the interval $[0, 2\pi]$.

   (a) Do the f_n converge uniformly to 0 on $[0, 2\pi]$?

   (b) Do the f_n converge pointwise to 0 on $[0, 2\pi]$?

   (c) Do the f_n converge strongly to 0 in L^2 (i.e., the L^2-norms $\|f_n\|_2 \to 0$ as $n \to \infty$)?

   (d) Do the f_n converge weakly to 0 in L^2 (i.e., $\int_0^{2\pi} f_n(x)g(x)dx \to 0$ for each g in $L^2([0, 2\pi])$?

   In each case, explain your reasoning.

8. Show without using theorems of Fourier analysis that every sequence $\{c_n\}_{n=-\infty}^{\infty}$ of complex numbers with $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$ is the sequence of Fourier coefficients of some function f in $L^2(-\pi, \pi)$.

Real Analysis, September 1988

1. For which positive real numbers p does the Lebesgue integral

$$\int_Q \frac{1}{(x^2+y^2)^p}\,dx\,dy$$

converge, where $Q = [0,1] \times [0,1]$? Justify your answer.

2. (a) If f is a non-negative Lebesgue integrable function on $[0,1]$ such that

$$\int_0^1 f(x)dx = 0,$$

prove that $f(x) = 0$ a.e.

(b) Show by an example that the assertion in (a) fails without the assumption that $f(x)$ is non-negative.

(c) If $f(x)$ is a real-valued Lebesgue integrable function on $[0,1]$ such that

$$\int_E f(x)dx = 0$$

for all measurable subsets E of $[0,1]$, show that $f(x) = 0$ a.e.

3. Assume $f_n \to f$ in $L^p(\mu)$ and $g_n \to g$ in $L^q(\mu)$, where p and q are in $(1,\infty)$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Show $f_n g_n \to fg$ in $L^1(\mu)$. Why are $f_n g_n$ $(n \geq 1)$ and fg in $L^1(\mu)$?

4. Define

$$f(t) = \sum_{n=1}^{\infty} \frac{e^{-nt}}{n^2}$$

for all positive numbers t. Answer the following questions, giving proofs of your assertions:

(a) On what intervals of $(0,\infty)$ is f uniformly continuous?

(b) At what points of $(0,\infty)$ is f differentiable?

(c) On what intervals of $(0,\infty)$ is f' uniformly continuous?

5. Suppose $f \in L^2(R)$. Let

$$g_n(k) = (2\pi)^{-1/2} \int_{|x|\leq n} e^{-ikx} f(x)dx$$

for all positive integers n and real k. Show that there is a function $g \in L^2(R)$ such that $g_n \to g$ in the metric of $L^2(R)$.

6. Let f and g be bounded, non-negative Lebesgue measurble functions on $(0,1)$. For any positive number y, let

$$E_y = \{x : x \in (0,1) \text{ and } g(x) \geq y),$$

and

$$\phi(y) = \int_{E_y} f(x)dx.$$

Prove that $\phi(y)$ is a non-increasing function of positive y and

$$\int_0^1 f(x)g(x)dx = \int_0^\infty \phi(y)dy.$$

Real Analysis, January 1989

1. Let $t_1, t_2, \ldots$ and t be real-valued, measurable functions on the real line. Let m be Lebesgue measure.

   (a) Prove or disprove this statement: if $f_n \to f$ a.e., then for any positive number $\epsilon > 0$,

   $$\lim_{n\to\infty} m(\{x : |f - f_n| > \epsilon\}) = 0.$$

   (b) Prove or disprove the converse of the statement in (a). (Hint: consider the indicator functions of the sets $[0,1]$, $[0,1/2]$, $[1/2,1]$, $[0,1/3]$, $[1/3,2/3]$, $[2/3,1]$, $[0,1/4], \ldots$)

2. Let a and b be positive numbers with $a + b = \frac{1}{4}$. Define a function $f(x)$ of real x to be:

   $$\begin{array}{ccc} 0 & \text{in}(-\infty, 0); & \\ a & \text{in} & [0, 1/4); \\ a+b & \text{in}[1/4, 1/2); & \\ x^2 & \text{in} & [1/2,); \\ 1 & \text{in} & [1, \infty). \end{array}$$

   Let μ be the Lebesgue-Stieltjes measure associated wiht f. Let μ be Lebesgue measure on the real line.

   (a) Evaluate

   $$\int_{-\infty}^{\infty} (x+1) d\mu.$$

   (b) Exhibit measures μ_1 and μ_2 on the real line such that $\mu = \mu_1 + \mu_2$ and $\mu_1 << m$ and $\mu_2 \perp m$.

   (c) Find a function g such that

   $$g = d\mu_1/dm$$

   a.e. $[m]$.

3. For each $r \in (0,1)$, define

   $$\mu_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int}, \ -\infty < t < \infty.$$

   (a) Prove that the infinite series converges uniformly in t for each fixed r.

   (b) Prove that for each fixed r,

   $$p_r(t) = \frac{1 - r^2}{1 - 2r\cos t + r^2}.$$

(c) Prove that for each fixed r,

$$\int_0^{2\pi} p_r(t)dt = 1.$$

4. Let μ be a finite measure on a set X, and let $f_1, f_2, \ldots$ and f be measurable functions on X such that $f_n \to f$ uniformly on X.

(a) Show that

$$\lim_{n\to\infty} \int_X |f(x) - f_n(x)| d\mu(x) = 0.$$

(b) Give an example to show that the conclusion of (a) may fall if $\mu(X) = \infty$.

Real Analysis, May 1989

1. (a) If μ is a finite measure on a space Ω and $1 \leq r \leq s < \infty$, show that $L^s(\Omega, \mu) \subseteq L^r(\Omega, \mu)$.

   (b) Prove or disprove that the assertion is true without the assumption that the measure is finite.

2. (a) Let $f_1, f_2, \ldots$ and f be continuous, real-valued functions on a closed and bounded interval $[a, b]$ such that $f_n \to f$ uniformly on $[a, b]$. Prove or disprove that $f_n^2 \to f^2$ uniformly on $[a, b]$.

   (b) What happens if $[a, b]$ is replaced by $(-\infty, \infty)$?

3. (a) Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $\mathcal{B}$ a sub-σ-field of $\mathcal{F}$. Show that if f is a real-valued function in $L^1(\Omega, \mathcal{F}, \mu)$, then there exists a $\mathcal{B}$-measurable function h on Ω such that

   $$\int_B f \, d\mu = \int_B h \, d\mu$$

   for all $B \in \mathcal{B}$.

   (b) Find the function h if Ω is $[0, 1]$ with Lebesgue measure $\mathcal{B}$ is the σ-field generated by the two intervals $[0, 1/2)$ and $[1/2, 1]$, and $f(x) = x^2$ for all x.

4. Let f and g be two functions in $L^2(-\infty, \infty)$.

   (a) Show that

   $$\lim_{x \to 0} \int_{-\infty}^{\infty} |g(t + x) - g(t)|^2 dt = 0.$$

   (b) Show that

   $$F(x) = \int_{-\infty}^{\infty} f(t) g(x - t) dt$$

   defines a continuous function on $(-\infty, \infty)$.

5. (a) Let $\Omega_1 = [0, 1]$ with Lebesgue measure μ_1, and let $\Omega_2 = [0, 1]$ with counting measure μ_2 (counting measure is the measure which assigns the number of points to any set). If f is the characteristic function on the diagonal $\{(x, x) : x \in [0, 1]\}$, show by evaluating both sides that

   $$\int_{\Omega_1} \left\{ \int_{\Omega_2} f \, d\mu_2 \right\} d\mu_1 \neq \int_{\Omega_2} \left\{ \int_{\Omega_1} f \, d\mu_1 \right\} d\mu_2.$$

   (b) Explain why this example does not contradict Fubini's theorem.

Real Analysis, September 1989

1. True or False. If true, give a reason; if false, give a counter-example. m denotes Lebesgue measure.
   (a) If E_n is a measurable subset in $\mathbb{R}$ and $E_n \downarrow E$, then $m(E_n) \downarrow m(E)$.
   (b) If E is a closed set in $\mathbb{R}$ and $m(E) = 0$, then E is nowhere dense.
   (c) If E is measurable in $\mathbb{R}$, then $m(\bar{E}) = m(E)$, where $\bar{E}$ denotes the closure of E.
   (d) If f is continuous and bounded on $\mathbb{R}$, then $\int_{\mathbb{R}}$, then $\int_{\mathbb{R}} f\,dm$ exists.

2. Let A be a Borel subset of $[0,1] \times [0,1]$. For each $x \in [0,1]$ and each $y \in [0,1]$, let

$$\begin{aligned} A_x &= \{y \in [0,1] : (x,y) \in A\}; \\ A^y &= \{x \in [0,1] : (x,y) \in A\}, \end{aligned}$$

   be the x and y sections of A respectively.
   (a) Show that $A_{1/2}$ is a Borel set in $[0,1]$.
   (b) If A_x is countable for almost all $x \in [0,1]$, show that for almost all y, $m(A^y) = 0$.

3. Let $v \in L^1(\mathbb{R})$.
   (a) Show that for any $f \in L^1(\mathbb{R})$,

$$\|v^* f\|_1 \le \|v\|_1 \|f\|_1,$$

   where $v^* f$ is the convolution.
   (b) Does $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converge to a continuous function on $\mathbb{R}$? Explain.
   (c) Is $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ the Fourier series of an $L^2[0, 2\pi]$ function? Explain.

4. If f is differentiable on $[0,1]$ with $f' \in L^2([0,1])$, show that

$$|f(x) - f(y)| \le |x-y|^{1/2} \left(\int_0^1 |f'|^2 dx \right)^{1/2}$$

   for $x < y$ in $[0,1]$.

5. Are the following two equations correct? Explain.
   (a) $\displaystyle \lim_{n\to\infty} \int_0^{2\pi} \frac{e^{inx}}{1+x} = 0,$
   (b) $\displaystyle \lim_{\epsilon \downarrow 0} \int_0^{\infty} \frac{e^{-x}}{1+\epsilon^2 x} = 1.$

Real Analysis, May 1990

1. Let $\{r_n\}$ be an enumeration of the rationals in $[0, 1]$, and let

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n |x - r_n|^{\frac{1}{2}}}.$$

   Show that

   (a) $f(x)$ is measurable;

   (b) the series converges to an $L^1([0, 1], dx)$-function;

   (c) the series converges pointwise a.e.

2. (a) Let $\mathcal{B}$ be a Banach space, λ a bounded linear functional on $\mathcal{B}$. Show that ker $\lambda \equiv \{x \in \mathcal{B} | \lambda(x) = 0\}$ is a closed linear subspace of $\mathcal{B}$.

   (b) For what values of p is λ a bounded linear functional on $L^p(\mathbb{R}, dx)$, where

$$\lambda(f) = \int_0^1 \frac{f(x)}{x^{\frac{1}{2}}}\, dx.$$

3. Suppose $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x + n) g(x) dx = 0.$$

4. Let f be absolutely continuous with $\frac{df}{dx}$ in $L^2(\mathbb{R})$. Prove that there exist constants L, $\alpha \geq 0$, such that

$$|f(x) - f(y)| \leq L|x - y|^{\alpha}, \; x, y \in \mathbb{R}.$$

5. (a) Explicitly sum the series

$$\sum_{n=0}^{\infty} \frac{e^{inx}}{2^n}.$$

   (b) Using Parseval, compute the integral

$$\int_0^{2\pi} \frac{\cos(1990x) dx}{2 - e^{ix}}.$$

Real Analysis, September 1990

1. (a) Prove that if f and g are real-valued functions on the reals, with f Lebesgue measurable and g continuous, then the composition $g \circ f$ is measurable.

   (b) Prove that if $\{f_n\}$ is a sequence of real-valued measurable functions on the reals

$$S_N \equiv \left\{ x \in \mathbb{R} | \sup_n f_n(x) > N \right\}$$

   is measurable.

2. (a) Prove that the x-axis in the two-dimensional Euclidean plane $\mathbb{R}^2$ has zero (two-dimensional) Lebesgue measure.

   (b) Let $g(x)$ be a continuous real-valued function on $[0,1]$. Let S be the graph of g in $\mathbb{R}^2$, i.e.,

$$S = \left\{ (x,y) \in \mathbb{R}^2 | y = g(x),\ 0 \le x \le 1 \right\}.$$

   Show, by covering S with rectangles, that S has zero two-dimensional Lebesgue measure.

3. Prove that if f, g are two L^2-functions on $\mathbb{R}$, then

$$\lim_{t \to 0} \int f(x)g(x+t)dx = \int f(x)g(x)dx.$$

4. Suppose f is continuous and bounded. Prove that

$$\lim_{N \to \infty} N \int_0^\infty f(x)e^{-Nx}dx = f(0).$$

5. Let $\{f_n\}$, f, g be $L^2[0,1]$-functions with $f_n \to f$ pointwise a.e.

   (a) Suppose $|f_n(x)| \le |x|^{-\frac{1}{3}}$ for each n. Prove

$$\lim_{n \to \infty} \int f_n(x)g(x)dx = \int f(x)g(x)dx.$$

   (b) Suppose instead that the f_n's satisfy

$$\int |f_n|^2 dx < 1990 \text{ for each } n.$$

   Show that if g is moreover bounded, then

$$\lim_{n \to \infty} \int f_n g\, dx = \int f g\, dx.$$

   Hint: Use Egoroff, regarding almost uniform convergence.

Real Analysis, January 1991

1. (a) Prove that if f is a real-valued measurable function, then so is $|f|$.

   (b) Prove that if for each y, $\{x|f(x) > y\}$ is Borel, then $f^{-1}(B)$ is a Borel set, for each Borel set B.

2. (a) Give an example of a non-negative sequence $\{f_n\}$ of integrable functions for which the inequality of Fatou's lemma is strict.

   (b) Prove or give a counter-example: If $\{f_n\}$ is a non-negative sequence of integrable functions on $\mathbb{R}$ which converges *uniformly* to an integrable function f,

$$\lim_{n\to\infty} \int f_n dx = \int f\, dx.$$

3. Show that $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges pointwise on $[-1, 1]$ but does not converge uniformly.

4. Let $f(x)$ be an integrable function such that

$$\int_0^\infty x|f(x)|dx < \infty.$$

   Prove that

$$\frac{d}{dt}\int_0^\infty \sin xt\, f(x)dx = \int_0^\infty x\cos xt\, f(x)dx.$$

5. Suppose $\{f_n\}$ is a sequence in L^p for some p, $1 \leq p < \infty$ with $f_n \to 0$ in L^p norm. Prove that $f_n \to 0$ in measure.

6. Let f be a uniformly continuous function in $L^p(\mathbb{R})$, for some p, $1 \leq p < \infty$. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

General Exam: Analysis
January 18, 1992

Name:

1. **(a)** Suppose f is Lebesgue measurable and satisfies

$$|\int fg dx| \leq \|g\|_{L^2(\mathcal{R})} \qquad (*)$$

for all L^2 functions g. Prove that f is in L^2. What is the L^2-norm of f?

(b) Suppose that we have the inequality $(*)$ only for functions g which are bounded and of compact support. Prove that f is in L^2.

2. **(a)** Let $\phi_1, \phi_2, \ldots, \phi_N$ be an orthonormal collection of functions in $L^2(\mathcal{R}, dx)$, and let $a_1, a_2, \ldots, a_N$ be real constants. Prove that

$$m\{x| \quad |\sum_{i=1}^{N} a_i \phi_i(x)| > \epsilon\} < \frac{1}{\epsilon^2} \sum_{i=1}^{N} a_i^2.$$

where m is Lebesgue measure.

(b) Let (X, μ) be a measure space with $\mu(X) = 1$, and let f be such that

$$\int d\mu\, f = 0, \qquad \int d\mu\, f^2 = 1.$$

Let μ^N be the product measure $\underbrace{\mu \times \cdots \times \mu}_{N}$ on $\underbrace{X \times \cdots \times X}_{N}$, and let

$$F(x_1, \ldots, x_N) = \frac{1}{N}[f(x_1) + \cdots + f(x_N)]$$

where $x_i \in i^{th}$ factor X in $X \times \cdots \times X$. Prove that

$$\mu^N(\{(x_1, \ldots, x_N) : |F(x_1 \cdots x_N)| > \epsilon\}) \leq \frac{1}{N\epsilon^2}.$$

3. Let

$$f_n(x) = \begin{cases} 0 & for \quad x \leq 0; \\ nx & for \quad 0 < x \leq \frac{1}{n}; \\ 1 & for \quad x > \frac{1}{n}. \end{cases}$$

Evaluate

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x)\, g\prime(x) dx$$

where $g(x)$ is a continuously differentiable function of compact support, such that $g(0) = 2$. Here, $g\prime$ is the derivative of g.

4. **(a)** Give an example of a continuous function which is not absolutely continuous on $[0, 1]$.

   (b) Let Λ be a bounded linear functional on $L^p(\mathcal{R}, dx)$, for some p, $1 \leq p < \infty$. Prove that

$$F(x) \equiv \Lambda(\chi_{[0,x]})$$

   is absolutely continuous. (Here $\chi_{[0,x]}(\cdot)$ is the characteristic function for the interval $[0, x]$.

5. **(a)** Give an example of a sequence of functions $\{f_n\}$ with $f_n,\ f \in L^1(\mathcal{R}, dx)$, $f_n \to f$ a.e., but f_n does not converge to f in L^1-norm.

   (b) Prove that if $f_n \to f$ a.e., and if $\|f_n\|_1 \to \|f\|_1$ ($\|\cdot\|_1$ is the L^1-norm), then $\|f_n - f\|_1 \to 0$. Hint: Consider

$$\int (|f_n| + |f| - |f - f_n|) dx.$$

General Exam: Analysis
January 23, 1993

Name:

1. Let $f(x,y)$ be defined on $J = [0,1)^2$ by the formula

$$f(x,y) = 1/(1-xy).$$

Prove that f is Lebesgue integrable on J, and letting λ be Lebesgue measure on J, show that

$$\int_J f d\lambda = \sum_{n\geq 1} \frac{1}{n^2}.$$

2. Let $\{a_n\},\ \{b_n\}$ be bounded sequences and suppose that

$$f_n(x) \equiv a_n \cos nx + b_n \sin nx$$

converges to zero, a.e. in $x,\ n \to \infty$. Prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Hint: Consider the $L^2[0,2\pi]$-norm of f_n.

3. Let $\phi(y) \geq 0$ be a bounded continuous on $[0,1]$, and define $g(x)$ by the formula

$$g(x) = \int_{[0,1]} \phi(y)/(1-xy)dy.$$

Prove that g is continuously differentiable on $[0,1)$, with

$$g'(x) = \sum_{n\geq 0} c_n x^n,$$

where

$$c_n = (n+1)\int_{[0,1]} \phi(y) y^{n+1} dy.$$

4. Let f be a bounded real-valued function on $[0,1]$ whose set of discontinuities is a set of Lebesgue measure zero.

   (a) Prove or give a counterexample to the statement that f is Lebesgue measurable.

   (b) Prove or give a counterexample to the statement that f is Borel measurable.

5. Suppose that

   (i) $\sum_{n=1}^{\infty} a_n$ is a convergent series (not necessarily absolutely convergent).

   (ii) $\{u_n(x)\}$ is a sequence of continuous functions.

   (iii) $0 \leq u_{n+1}(x) \leq u_n(x)$.

   (iv) $u_n(x) \to 0,\ n \to \infty$ and convergence is uniform.

   Prove that $\sum_{n=1}^{\infty} a_n u_n(x)$ converges to a continuous function.

   Hint: Let $s_n = \sum_{m=1}^{n} a_m$. Then $a_n = s_n - s_{n-1}$ and so

$$\sum_{n=1}^{N} a_n u_n = \sum_{n=1}^{N} (s_n - s_{n-1}) u_n.$$

   Sum by parts.

Real Variable General Exam **April 3, 1993**

1. Give conditions on the sequence a_n such that

$$S(x) = \sum_{n=1}^{\infty} \frac{a_n}{x + \frac{1}{n}}$$

is (a) continuous, and (b) continuously differentiable on $[0, 1]$.

2. Let μ be a positive, *non-atomic*, Borel measure on $[0, 1]$ of total mass one. Prove that the interval $[0, c]$ has μ measure for $\frac{1}{2}$ for some c.

3. Give an example of a *bounded* function $f(x)$ on $[0, 1]$ such that

   (a) $f(x)$ is not measurable;

   (b) $f(x)$ is measurable, but not Riemann integrable;

   (c) $f(x)$ is differentiable at all points, but $f'(x)$ is not continuous.

4. Give an example of a subset S of [0,1] such that

   (i) S is countable;

   (ii) every point of S is isolated; and

   (iii) the boundary ∂S of S has positive measure.

5. Let $f(x)$ be measurable on $[0, \infty)$ and define

$$F(s) = \int_{\infty}^{0} \frac{f(x)}{(1 + sx)^2}\, dx.$$

   (a) If $\frac{f(x)}{x} \in L_1(0, \infty)$, prove that $F(s)$ is finite a.e. and is in $L_1(0, \infty)$.

   (b) Prove that if $f(x) \geq 0$ and $F(s)$ is finite and bounded on $[0, \infty)$, then $f(x) \in L_1$.

   (c) Assume that $f(x)$ is continuous on $0 \leq x$, and that $f(\infty) = \lim_{x \to \infty} f(x)$ exists. Find

$$\lim_{x \to 0} s\, F(s),$$

   and

$$\lim_{x \to \infty} s\, F(s).$$

Probability and Real Analysis General Examination
2 hours Fall 1993

1. For real $x > 0$, set
$$f(x) = \sum_{n=1}^{\infty} e^{-n^2 x}.$$
   (a) Is $f(x)$ bounded on $(0, \infty)$?
   (b) Is $f(x)$ integrable on $(0, \infty)$?
   (c) Is $f(x)$ continuous on $(0, \infty)$?

2. Let $\{X_n\}$ be i.i.d. with common exponential distribution
$$P\{X_n > \lambda\} = e^{-\lambda} \text{ for } \lambda \geq 0.$$
   Show that
$$P\left\{\limsup_{n\to\infty} \frac{X_n}{\log n} \geq 1\right\} = 1.$$

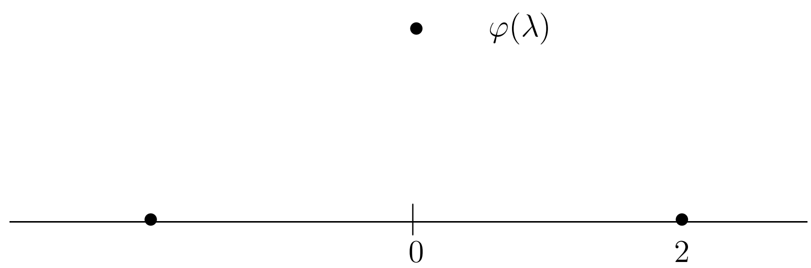
3. Suppose $\{X_n\}$ is an i.i.d. sequence with an unknown non-degenerate distribution function $F(x)$. Let
$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} 1_{\{x_j \leq x\}}$$
   be the so-called empirical distribution function of $\{x_n, \ldots, x_n\}$.

   Discuss whether or not
   (a) $P\{\lim_{n\to\infty} F_n(x) = F(x)\} = 1$, and
   (b) For large n, $F_n(x) - F(x)$ is approximately normally distributed.

4. The graph of a characteristic function

$$\varphi(\lambda) = Ee^{i\lambda x}$$

is drawn below.



(a) Find $P\{X = 0\}$.

(b) Does X have an absolutely continuous distribution?

(c) Is $E|X| < \infty$?

5. Let $\{T_n\}$ be i.i.d. positive random variables with a common distribution function $F(t)$. Suppose $\lim_{t \downarrow 0} \frac{F(t)}{t} = \lambda$ exists, and define

$$\begin{aligned} Z_n &= \text{number of } j \text{ with } 1 \le j \le n \text{ for which } 0 \le T_j \le \frac{1}{n} \\ &= \sum_{j=1}^{n} 1_{\{T_j \le \frac{1}{n}\}}. \end{aligned}$$

Show that $\{Z_n\}$ converges in distribution and find the limiting distribution.

Analysis General Exam

January 22, 1994

1. True or False (Give a proof or a counterexample.)

   (a) If f is a non-negative continuous function on $\mathbb{R}$ and
   $$\int f \, dm < \infty$$
   (m is Lebesgue measure) then
   $$\lim_{x \to \infty} f(x) = 0.$$

   (b) If f is equal a.e. to a continuous function g on $[a, b]$, then f is continuous a.e.

2. Do the following limits exist? Why or why not?
   $$\lim_{\epsilon \searrow 0} \int_\epsilon^1 \sin\left(\frac{1}{x}\right) dx$$
   $$\lim_{N \to \infty} \int_0^N \sin x^2 dx$$

3. Let $\{q_i > 0\}, \ i = 1, 2, \ldots$ be a sequence of positive numbers such that
   $$\sum_i q_i < \infty,$$
   $\{\mu_i\}$ another arbitrary sequence of real numbers. Let
   $$f_N(x) = \sum_{i=1}^N \frac{q_i}{x - \mu_i}, \quad x \in \mathbb{R}.$$
   It is a fact (do not show this!) that if $\lambda > 0$,
   $$m\{x | \, f_N(x) > \lambda\} = m\{x | \, f_N(x) < -\lambda\} = \frac{1}{\lambda} \sum_{i=1}^N q_i$$

   (a) Prove that the sequence $\{f_N(x)\}$ forms a Cauchy sequence in measure.

   (b) For $0 \leq \rho < 1$ and
   $$I_N = \{x | \, f_N(x) > 1\}$$
   compute
   $$\int_{I_N} f_N^\rho dx,$$
   using the fact above.

4. Let $\{p_n(x)\},\ n = 0, 1, 2, \ldots$ be a sequence of real polynomials with the following properties:

   (i) $p_n(x)$ is a polynomial of degree n (and the coefficient of x^n is not zero).

   (ii)
$$\begin{aligned} \int_{-1}^{1} p_n(x) p_m(x) dx &= 1 \quad \text{if } n = m \\ &= 0 \quad \text{otherwise} \end{aligned}$$

   Show that $\{p_n(x)\}$ constitutes a complete orthonormal basis in $L^2([-1, 1], dx)$.

5. Let m denote two-dimensional Lebesgue measure on the plane $\mathbb{R}^2$. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be continuous and let $G = \{(x, y) \in \mathbb{R}^2 |\ y = f(x)\}$.

   (a) Show that G is closed, hence measurable.

   (b) Use Fubini's theorem to show that $m(G) = 0$.

6. Show that if f is measurable on $[a, b]$ and $f'(x)$ exists a.e. on (a, b), then $f'(x)$ is measurable. Suppose, moreover, that f is continuous. Does f satisfy

$$f(b) - f(a) = \int_a^b f'(x) dx?$$

Analysis General Exam, Pt. 1

September 2, 1994

1. Let $f(x)$ be the trigonometric series
$$f(x) = \sum_{n\geq 1} \frac{\cos nx}{n^p}$$
   (a) Show that for $p > 1$, $f(x)$ is continuous.
   (b) Prove that for $p > 2$,
$$f'(x) = -\sum_{n\geq 1} \frac{\sin nx}{n^{p-1}}$$

2. Let
$$\begin{aligned} f(x) &= e^{-\frac{1}{x}} \quad x > 0 \\ &= 0 \qquad x \leq 0 \end{aligned}$$
   (a) Show that f is continuously differentiable on $\mathbb{R}$.
   (b) Show more generally that f is infinitely differentiable on $\mathbb{R}$.

3. Suppose that $g(x) \geq 0$ is a twice continuously differentiable function defined on $\mathbb{R}$ with $g(0) = g(1) = 0$. *Prove* that there is a point $x \in (0, 1)$ such that
$$g''(x) \leq 0.$$
   (Pictures won't suffice.)

4. Suppose F is defined by
$$\begin{aligned} F(x) &= 0 \quad x < 0 \\ &= x^2 \quad 0 \leq x < 1 \\ &= n^2 \quad n \leq x < n+1 \end{aligned}$$
   Compute
$$\int \frac{e^{-x}}{x} dF(x).$$

5. Suppose $\{f_n\}$ is a sequence of integrable functions on $\mathbb{R}$ such that if

$$a_n \equiv \int |f_n| dx$$

and

$$\sum_n a_n < \infty.$$

Prove that

$$\int \sum_n f_n(x) dx = \sum_n \int f_n(x) dx.$$

6. Suppose that $\{f_n\}$ is a sequence of non-negative functions on $[0, 1]$ such that

$$\lim_{n\to\infty} \int f_n dx = 0.$$

Compute

$$\lim_{n\to\infty} \int_{[0,1]} e^{-f_n(x)} dx,$$

and justify your answer.

7. Let $f(x), g(x)$ be Lebesgue measurable functions. Show that

$$h(x, y) = f(x),$$

and

$$k(x, y) = f(x)g(y)$$

are measurable with respect to 2-dimensional Lebesgue measure.

General Exam: Real Analysis
Winter 1996

1. (a) Prove the Weierstrass M-test for continuity of an infinite series of functions: Let $\{f_n(x)\}$ be a sequence of continuous functions on some set S, let and $\{m_n\}$ be a sequence of non-negative constants such that $|f_n(x)| \leq m_n$ on S, for each n. If $\sum m_n$ is finite then

$$F(x) \equiv \sum_{n=0}^{\infty} f_n(x)$$

converges to a continuous function.

   (b) Is

$$F(x) = \sum_{n=0}^{\infty} n^{1996} e^{-n^2 x}$$

continuous for $x > 0$?

2. Show that if the real-valued function f is differentiable on $[a, b]$, then $f' = df/dx$ satisfies the conclusion of the intermediate-value theorem, i.e., that given an intermediate value λ between $f'(a)$ and $f'(b)$, then there is a $c \in [a, b]$ such that $f'(c) = \lambda$. To do this:

   (a) First show that if $f'(a) < 0$ and $f'(b) > 0$, then there exists a point $c \in [a, b]$ such that $f'(c) = 0$.

   (b) Prove the general case of the intermediate value theorem by considering $g(x) \equiv f(x) - \alpha x$, for suitable α.

3. Let $A = \{x \in [0, 1] : x = \sum_{n=1}^{\infty} a_n 10^{-n}$ and $a_n = 2$ or $7\}$. Prove or disprove:

   (1) A is countable

   (2) A is closed

   (3) A is open

   (4) A is dense in $[0, 1]$

   (5) A is Lebesgue measurable.

4. (a) Suppose that f is a non-negative Lebesgue measurable function on $[0,1]$, and let $\{x_n\}$ be a sequence of points, $x_n > 0$, converging to 0. Show that if

$$\lim_{n\to\infty} \int_{[x_n,1]} f\,dm$$

exists finitely, then $\int_{[0,1]} f\,dm$ exists. Here, m is Lebesgue measure.

(b) Show that if f is not non-negative, the conclusion need not hold.

5. Is there a Lebesgue measurable set $E \subseteq [0,1]$ such that

$$m(E \cap [a,b]) = \frac{1}{2}(b-a)$$

for all $0 \le a \le b \le 1$? Here, m is again Lebesgue measure.

6. Let f be a non-negative real-valued function on the real line $\mathbf{R}$, and in $L^p(\mathbf{R}, dm)$ for all $1 \le p < \infty$.

(a) Show that

$$g(\lambda) \equiv m(\{x : f(x) > \lambda\}) \le \frac{C_p}{\lambda^p}$$

for each $1 \le p < \infty$, for some constant C_p independent of λ, $\lambda > 0$.

(b) Show that g is continuous from the right. Is g necessarily continuous from the left? Justify your answer.

(c) Express

$$\int_{\lambda \ge 0} g(\lambda)\lambda^r\,d\lambda,$$

$r > -1$, in terms of the L^p-norms of f.

7. For any subset S of the plane $\mathbf{R}^2$, define its horizontal (resp. vertical) sections as

$$S_y \equiv \{x : (x,y) \in S\} \ , S^x \equiv \{y : (x,y) \in S\}.$$

Let $\mathcal{A}$ be the collection of sets $\{S\}$ such that for each S, all of its horizontal and vertical sections are Borel subsets of the real line $\mathbf{R}$.

(a) Show that $\mathcal{A}$ is a σ-algebra containing the Borel subsets of the plane.

(b) Let $f(x,y)$ be a Borel measurable function on the plane. Show that for each fixed y,

$$g_y(x) \equiv f(x,y)$$

is a Borel function on the real line.

General Exam: Real Analysis, Fall, 1997

Note In all problems below, measure refers to Lebesgue measure on $\mathbf{R}^n$ unless otherwise noted.

1. (a) Show that if $\{f_N\}$ is a sequence of real-valued continuous functions converging uniformly to a function f on a topological space X, then f is continuous.
   Assume now that $\{a_n\}$ is a monotone sequence of real numbers with
   $$\lim_{n\to\infty} a_n = 0$$
   (b) Show that
   $$\left|\sum_{n=N}^{\infty}(-1)^n a_n\right| \leq |a_N|.$$
   (c) Use parts (a) and (b) to show that
   $$\sum_{n=1}^{\infty}(-1)^n a_n x^n$$
   is continuous for $x \in [0,1]$.

2. Show that $F(x)$ has a right derivative at $x = 0$, where
   $$F(x) = \int_0^{\infty} e^{-t^2 - xt^3} dt.$$

3. Give an example of each of the following and <u>explain</u>:
   (a) An L^1-Cauchy sequence of functions which does not converge pointwise anywhere. Show that your sequence has a subsequence which does converge pointwise, almost everywhere.
   (b) A closed measurable set (with respect to Lebesgue measure) having no interior and having positive measure.
   (c) A sequence of functions $\{f_n\}$ in $L^1(\mathbf{R})$ going to zero pointwise almost everywhere $n \to \infty$, but such that
   $$\lim_{n\to\infty} \int f_n dx \neq 0.$$
   (d) A continuous monotone function $f(x)$, differentiable almost everywhere, such that
   $$f(b) - f(a) \neq \int_a^b f'(x)dx.$$

4. Let $f(x) = |x|$, $x \in \mathbf{R}$, and let $\mathcal{A}$ be the smallest σ-algebra for which f is measurable. Characterize $\mathcal{A}$; In particular, explain the relationship between $\mathcal{A}$ and Borel sets on $\mathbf{R}$? If f is replaced by $f = \sin x$ characterize the σ-algebra in this case.

5. (a) Suppose that f is in $L^1(\mathbf{R})$. Show that

$$\lim_{t \to 0} \int |f(x) - f(x+t)| dx = 0.$$

(b) Is it necessarily true that if $f \in L^\infty(\mathbf{R})$, then

$$\lim_{t \to 0} \|f(\cdot + t) - f(\cdot)\|_\infty = 0.$$

Here, $\|g(\cdot)\|_\infty$ denotes the infinity norm of g. Prove or give a counter example.

6. Let $\alpha > 1/2$ be fixed and define the sequence of functions $\{f_N\}$ by

$$f_N(x) \equiv \sum_{n=1}^{N} \frac{\cos(nx)}{n^\alpha}.$$

(a) Show that the sequence $\{f_N\}$ converges in $L^2([0, 2\pi])$, $N \to \infty$.

(b) Show generally that a sequence $\{g_N\}$ of functions which converges in $L^2(X, \mu)$ with (X, μ) a finite measure space, also converges in $L^1(X, \mu)$. Conclude that the sequence $\{f_N\}$ above also converges, regarded as a sequence in $L^1([0, 2\pi])$.

7. Let $g(x) \geq 0$ be a real-valued continuous function with support in $[-1, 1]$ and such that

$$\int g(x) dx = 1.$$

Define g_N by $g_N(x) \equiv N g(Nx)$, for $N = 1, 2, \cdots$.

(a) Show that g_N has integral 1, and that its support is in $[-1/N, 1/N]$.

(b) Let $f(x)$ be a continuous function. Then compute

$$\lim_{N \to \infty} \int g_N(x - y) f(y) dy$$

and justify your computation.

8. (a) Prove that if $\{f_n\}$ is a sequence of functions in $L^p(\mathbf{R})$ with $\|f_n\|_p \to 0$, $n \to \infty$, then $\{f_n\}$ goes to zero in measure. (Here, $\|\cdot\|_p$ denotes the p-norm.)

(b) Let $f \geq 0$ be a function on a σ-finite measure space X with measure μ such that

$$\mu\{x : f(x) > \lambda\} = \frac{1}{1 + \lambda^2}.$$

Compute the L^1-norm of f. Is f in L^2?

General Exam: Real Analysis, Winter, 1998

1. (a) Let $f(x)$ be a continuous function defined on the reals $\mathbf{R}$ and such that

$$\lim_{x\to\infty} f(x) \text{ and } \lim_{x\to-\infty} f(x)$$

both exist. Show that f is uniformly continuous on $\mathbf{R}$.

(b) Is $f(x) = \sin x^2$ uniformly continuous on $\mathbf{R}$? Prove your claim.

2. Suppose that $g(x) \geq 0$ is a twice continuously differentiable function defined on $\mathbf{R}$ with $g(0) = g(1) = 0$. *Prove* that there is a point $x \in (0,1)$ such that

$$g''(x) \leq 0.$$

3. Show that for $f \in L^1(0,\infty)$,

$$\lim_{k\to\pm\infty} \int_0^\infty e^{ik\cos\pi x} f(x)\, dx = 0.$$

Hint: You could integrate by parts using

$$\frac{i}{k\pi \sin \pi x} \frac{d}{dx} e^{ik\cos\pi x} = e^{ik\cos\pi x},$$

if f is a C^∞-function with compact support S disjoint from the integers, i.e., $S \subset (0,\infty) \setminus \{\text{integers}\}$.

4. Suppose that $\{f_n\}_n^\infty$ is a sequence of non-negative measurable functions on a finite measure space (X,μ). Suppose moreover that

$$\sum_{n=1}^\infty \int f_n(x)\, d\mu(x) < \infty.$$

Show that except for a set of measure zero, $f_n(x) \geq 1$ occurs for only finitely many n.

5. Suppose that $\{f_n\}_n^\infty$ is a sequence of non-negative integrable functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_{[0,1]} f_n\, dx = 0.$$

Compute

$$\lim_{n\to\infty} \int_{[0,1]} e^{-f_n}\, dx,$$

and justify your answer.

6. Let f be a continuous function, periodic with period 2π. Show that

$$\frac{1}{2\pi} \lim_{\rho\nearrow 1} \int_0^{2\pi} \frac{(1-\rho^2) f(\varphi)}{1+\rho^2 - 2\rho\cos(\theta-\varphi)}\, d\varphi = f(\theta)$$

with the convergence uniform in θ. You may assume without proof that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1-\rho^2)}{1+\rho^2 - 2\rho\cos(\theta-\varphi)}\, d\varphi = 1 \text{ for } \rho \in [0,1).$$

General Exam: Real Analysis, Fall 1998

1. For $x \in [-\pi, \pi]$, consider the infinite sum

$$S(x) = \sum_{n=0}^{\infty} 2^{-n} \cos(2^n x).$$

   (a) Show that the series $S(x)$ converges for all x and defines a continuous function.

   (b) Evaluate the integral

$$\int_{-\pi}^{\pi} S(x)dx$$

   and justify your reasoning.

   (c) Find the location of the maximum of the function $S(x)$ on the interval $[-\pi, \pi]$, and show that it is unique.

2. (a) Suppose $f \in L^2(0, \infty)$ is such that

$$\int_a^b f(x)dx = 0$$

   for each $0 < a < b < \infty$. Show, without using the fundamental theorem of calculus, that $f = 0$ a.e.

   (b) Let $f \in L^2(0, \infty)$ be such that for each $b > 0$,

$$\int_a^b e^x f(x)dx = 1 - \frac{1}{1+b}.$$

   Determine $f(x)$ and justify your procedure.

3. Let $\mathcal{B}_1$ be the Borel σ-field on R^1 and let $\mathcal{B}_2 = \mathcal{B}_1 \times \mathcal{B}_1$ be the product σ-field on R^2. Using the definition of a product σ-field, prove that $D \in \mathcal{B}_2$ where D is the unit disc $D = \{(x, y) \in R^2 : x^2 + y^2 < 1\}$.

4. Let

$$F(x) = \int_x^{\infty} e^{-t^2} dt.$$

   Find

$$\int_0^{\infty} F(x)dx,$$

   justifying all steps.

5. Evaluate

$$\lim_{t \to 0} \int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} \sin(xt)dx.$$

   Justify all steps.

General Exam: Real Analysis Fall 1999

1. For $f \in L^1(\mathbf{R}, dx)$, and $r > 0$, define

$$A_r f(x) = \frac{1}{2r} \int_{B(r,x)} f(y) dy.$$

Here, $B(r,x)$ is an interval of radius r centered about x.

(a) Show that the function $A_r f(x)$ is continuous in x and r.

(b) Show that the operation A_r is a contraction in the sense that $\|A_r f\|_1 \leq \|f\|_1$ for all f in L^1. (It may be helpful to think of A_r as a convolution.)

(c) Show that if f is a continuous function, then

$$\lim_{r \to 0} A_r f(x) = f(x).$$

(d) Using (b) and (c), show that if f is in L^1, then

$$\lim_{r \to 0} \|A_r f - f\|_1 = 0.$$

2. (a) Let $\{f_n\}$ be a sequence of real-valued functions on $[0,1]$, measurable with respect to Lebesgue measure μ, converging point-wise to the function f. Show that the sequence converges in measure, i.e., that for each $\epsilon > 0$

$$\lim_{n \to \infty} \mu\{x : |f_n(x) - f(x)| \geq \epsilon\} = 0.$$

(b) Suppose that $\{f_n\}$ converges in measure to f, again for $x \in [0,1]$. Show that

$$\lim_{n \to \infty} \int_{[0,1]} \cos(f_n(x)) dx = \int_{[0,1]} \cos(f(x)) dx$$

3. (a) Suppose that $\{x_n\}$, x are vectors in a Hilbert space $\mathcal{H}$, such that,

$$\lim_{n \to \infty} \langle x_n, y \rangle = \langle x, y \rangle \text{ for all } y \in \mathcal{H}.$$

Show that x is in the closure of the subspace of $\mathcal{H}$ spanned by the x_n's.

(b) If $\{x_n\}$ is an orthonormal sequence in $\mathcal{H}$, show that

$$\lim_{n \to \infty} \langle x_n, y \rangle = 0, \text{ for each } y \in \mathcal{H}.$$

(c) If $\{x_n\}$ is an orthonormal sequence in $\mathcal{H}$ and

$$y_n = x_{n+1} - x_n \text{ for each } n$$

show that if z is orthogonal to y_n for each n, then $z = 0$.

4. Let f be a non-negative function in $L^1([0,1], dx)$. Suppose that for every positive integer $n = 1, 2, \ldots$,

$$\int_{[0,1]} f^n dx = \int_{[0,1]} f dx.$$

Show that f must be equal to the characteristic function χ_E a.e., for some measurable set $E \subset [0,1]$.

5. Let $f(x) = |x|$, $x \in \mathbf{R}$. Let $\mathcal{A}$ be the *smallest* σ-algebra of $\mathbf{R}$ with respect to which f is measurable.

(a) Give a characterization of $\mathcal{A}$, i.e., describe the measurable sets. Characterize briefly the measurable functions with respect to $\mathcal{A}$. Hint: Given, say, an open interval U, what does $f^{-1}(U)$, which must be in $\mathcal{A}$, look like?

(b) Let μ and ν be two measures on $\mathcal{A}$, defined by

$$\mu(A) = \int_A e^{-x^2} dx, \ A \in \mathcal{A}$$

$$\nu(A) = \int_A e^{-x^2+x} dx, \ A \in \mathcal{A}$$

Show that μ is absolutely continuous with respect to ν, and compute the Radon-Nikodym derivative $d\mu/d\nu$. (Recall that this derivative should be $\mathcal{A}$-measurable.)

Real Analysis Exam, September 2000

1. a) Show that

$$L = \lim_{N\to\infty} \int_1^N x^{-\varepsilon} \cos x dx \tag{0.1}$$

exists for any $\varepsilon > 0$. Hint: integrate by parts.

b) Find conditions on $\alpha > 0$ and $\beta > 0$ so that

$$I(\alpha, \beta) = \int_0^\infty x^\alpha \cos{(x^\beta)} dx \tag{0.2}$$

exists as an (improper) Riemann integral.

2. Suppose that

$$\sum_{n=1}^\infty a_n \tag{0.3}$$

is an absolutely convergent series, and that $\{f_n\}$, $n = 1, 2, \cdots$ is a sequence of functions in $L^2(\mathbf{R}, dx)$ (or for that matter, any L^2 space) with L^2-norms uniformly bounded, $\|f_n\|_2 \leq 1$, for each n. Show that

$$\sum_{n=1}^\infty a_n f_n(x) \tag{0.4}$$

converges *pointwise* almost everywhere.

3. Let

$$f(x) = \inf_\alpha f_\alpha(x) \tag{0.5}$$

where $\{f_\alpha\}$ is a collection of continuous real-valued functions on $\mathbf{R}$. (The index α may run over an uncountable set.)

a) Show that $f(x)$ is upper-semicontinuous, i.e.,

$$S_t \equiv \{x : f(x) < t\} \tag{0.6}$$

is open for each t.

b) Is $f(x)$ Borel measurable? Why?

c) Suppose instead that each f_α is measurable, and that the index α runs over just a *countable* set. Show using the definitions of measurable functions and sets that $\inf_\alpha f_\alpha$ is measurable.

4. Suppose that $f(x)$ is a real-valued continuous function on $(0, 1)$ which is moreover *convex*,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{0.7}$$

$0 \leq \lambda \leq 1$. Recall that geometrically, this means that the graph of f lies below any cord of f so that the difference quotient $(f(y) - f(x))/(y - x)$ is monotone in x and y. (You don't need to prove this but it might be helpful to draw a picture.)

a) Show that $f(x)$ has a right derivative at x, for all $x \in (0, 1)$ which is uniformly bounded on compact intervals contained in $(0, 1)$. (Similarly, f has a left derivative, but you need not show this.)

b) Show that $f(x)$ is absolutely continuous. To do this it suffices to show that

$$f(b) - f(a) = \int_a^b \frac{d_r f(x)}{dx} dx \tag{0.8}$$

for any interval $[a, b] \subset (0, 1)$, ($\frac{d_r f}{dx}$ denotes the right derivative).

5. Show that

$$\lim_{n\to\infty} \sqrt{n} \int_0^{\pi/2} \cos^n x dx = \int_0^\infty \exp(-x^2/2) dx. \tag{0.9}$$

To show this, do the following:

a) For $0 < \delta \leq \pi/2$, show that

$$\lim_{n\to\infty} \sqrt{n} \int_\delta^{\pi/2} \cos^n(x) dx = 0. \tag{0.10}$$

b) show that for $\delta > 0$ sufficiently small,

$$1 - \frac{1}{2}x^2 \leq \cos x \leq e^{-x^2/2} \tag{0.11}$$

for $|x| \leq \delta$. You can do this just with power series or, more elegantly, by noting that

$$\cos x \leq 1 \leq \sec^2 x; \tag{0.12}$$

Integrate this inequality from 0 to x, then integrate again to get the desired inequalities (for $0 \leq x \leq \pi/2$).

c) Change variables; set $y = \sqrt{n}x$ and consider the integral

$$\sqrt{n} \int_0^\delta \cos^n(x) dx = \int_0^{\delta\sqrt{n}} \cos^n(y/\sqrt{n}) dy. \tag{0.13}$$

1. Suppose that $\{f_j\}_{j=1,2,\cdots}$ is a sequence of real-valued continuous functions on $[0,1]$ which are uniformly bounded, i.e., there is a constant c such that

$$|f_j(x)| \leq c$$

for all j and the functions are equi-continuous in the sense that for each $\epsilon > 0$, there is a $\delta > 0$ independent of j such that

$$|f_j(x) - f_j(y)| < \epsilon$$

for $|x - y| < \delta$. Suppose moreover that

$$\lim_{j\to\infty} f_j(x) = f(x)$$

exists for all $x \in [0,1]$. Prove that

(a) the convergence is uniform in $x \in [0,1]$.

(b) $f(x)$ is continous.

2. Suppose that $g(t)$ is a (complex-valued) integrable function on $[0,\infty)$, such that

$$\begin{aligned} \int_0^N t|g(t)|dt &\leq N^{1/2} \\ \int_N^\infty |g(t)|dt &\leq N^{-1/2}. \end{aligned}$$

for $0 < N < \infty$. Let $f(x)$ be defined,

$$f(x) \equiv \int_0^\infty e^{-itx} g(t)dt.$$

Show that $f(x)$ is Hölder continuous in the sense that

$$|f(x) - f(y)| \leq c|x-y|^\beta$$

for suitable constants c and β. To do this obtain estimates on

$$\int_0^N (e^{-itx} - e^{-ity})g(t)dt, \quad \int_N^\infty (e^{-itx} - e^{-ity})g(t)dt$$

and optimize your choice of N.

3. Show from the definition of measurable functions:

   (a) If $f(x)$ and $g(x)$ are (real-valued) Borel measurable functions defined on the real line $\mathbf{R}$, then their composition $f \circ g(x)$ is measurable.

   (b) If $\{f_j\}$ is a sequence of (real-valued) Borel measurable functions, then

$$f(x) \equiv \liminf f_j(x)$$

   is measurable.

4. In the following, let $\{f_j(x)\}_{j=1,2,\dots}$ be an $L^1[0,1]$-Cauchy sequence.

   (a) Show by an example, that $\{f_j(x)\}$ need not converge pointwise.

   (b) Show that there is a subsequence of $\{f_j(x)\}$ which does converge pointwise a.e.

   (c) Show that given $\epsilon > 0$ there is an N such that if $i, j \geq N$, then

$$m\{x : |f_j(x) - f_i(x)| \geq \epsilon\} < \epsilon.$$

   Here, $m\{\cdot\}$ refers to Lebesgue measure.

5. Suppose that $g(x)$ is a measurable and Lebesgue-integrable function (not necessarily positive) such that

$$\int_0^\infty g(t)dt = 1,$$

and suppose also that $f(x)$ is a bounded measurable function having a limit L for $x \to \infty$. Determine

$$\lim_{N\to\infty} \frac{1}{N} \int_0^\infty g(x/N) f(x) dx = \lim_{N\to\infty} \int_0^\infty g(x) f(Nx) dx$$

and prove your result.

6. Let $f(x)$ be an $L^2(\mathbf{R})$-function, and let f_h denote its translate, $f_h(x) = f(x+h)$. Show that

$$\lim_{h\to 0} f_h(x) = f(x)$$

in an L^2 sense.

7. Let $\{f_j(x)\}_{j=1,2,\dots}$ be a sequence of $L^2[0, 2\pi]$ functions, with the property that the Fourier coefficients

$$\hat{f}_j(n) \equiv \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f_j(x) dx,$$

$n \in \mathbf{Z}$ satisfy the following estimates:

(a) There exists a non-negative sequence $\{c(n)\} \in \ell^2(\mathbf{Z})$ (square-summable sequences) such that,

$$|\hat{f}_j(n)| \leq c(n)$$

for all $j \in \mathbf{Z}$, and

(b) for each n,

$$\hat{f}_\infty(n) \equiv \lim_{j\to\infty} \hat{f}_j(n)$$

exists.

Show that the sequence $\{f_j(x)\}$ converges in an L^2 sense.

General Exam: Real Analysis, Winter, 2003

1.a) Prove the well-known fact: Let $\{f_n(x)\}$ be a sequence of real-valued continuous functions on a topological space X which is uniformly Cauchy in the sense that for each $\epsilon > 0$, there is an N independent of $x \in X$ such that

$$|f_n(x) - f_m(x)| < \epsilon$$

for n, $m \geq N$. Then

$$f(x) \equiv \lim_{n\to\infty} f_n(x)$$

exists and is continuous.

b) Use your result above to prove the Weierstrass M-test: Let $\{g_n\}$ be a sequence of real-valued continuous functions defined on a topological space X. Assume moreover that each g_n satisfies a bound with $|g_n(x)| \leq m_n$, with $\sum m_n < \infty$. Then

$$g(x) \equiv \sum_{n=1}^{\infty} g_n(x)$$

exists and is continuous.

2. Suppose that $\{f_n(x)\}$ is a sequence of real-valued Lebesgue measurable functions which converge pointwise a.e. to a function $f(x)$. Suppose moreover that $f_n(x) \leq f(x)$ for all n and almost every x, and that there exists a constant $C < \infty$ independent of n such that

$$\int_{\mathbf{R}} e^{f_n(x)}\, dx \leq C.$$

Show that $e^{f(x)}$ is integrable, and that

$$\lim_{n\to\infty} \int_{\mathbf{R}} e^{f_n(x)}\, dx = \int_{\mathbf{R}} e^{f(x)}\, dx.$$

3. For any subset S of the plane $\mathbf{R}^2$, define its horizontal (resp. vertical) sections as

$$S_y \equiv \{x : (x, y) \in S\}\ , S^x \equiv \{y : (x, y) \in S\}.$$

Let $\mathcal{A}$ be the collection of sets $\{S\}$ such that for each S, all of its horizontal and vertical sections are Borel subsets of the real line $\mathbf{R}$.

(a) Show that $\mathcal{A}$ is a σ-algebra containing the Borel subsets of the plane.

(b) Let $f(x,y)$ be a Borel measurable function on the plane. Show that for each fixed y,

$$g_y(x) \equiv f(x,y)$$

is a Borel function on the real line.

4. Let $f(z)$ (z a complex number) be defined by the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}$$

Using the Schwarz inequality, show that

a)

$$|f(z)| \leq \frac{1}{\sqrt{1-a^2}} e^{|z|^2/2a^2}$$

for $0 < a < 1$.

b) For fixed r let

$$F_r(\theta) = f(re^{i\theta})$$

Compute

$$\int_0^{2\pi} |F_r|^2(\theta) d\theta$$

in closed form.

5. a) Let $\{c_n\}$ be a sequence of real numbers, converging to c, $|c| < \infty$. Compute

$$\lim_{a \uparrow 1} (1-a) \sum_{n=1}^{\infty} c_n a^n$$

and justify your computations.

b) Let $\{f_n\}$ be a sequence of $L^1(dx)$ functions which converge in L^1 to a function f. Compute

$$\lim_{a \uparrow 1} (1-a) \int_{\mathbf{R}} \sum_{n=1}^{\infty} a^n f_n(x) dx$$

and justify your computations.

6. Let $\mathcal{H}$ be the Hilbert space of 2π-periodic functions on the real line with inner product $\langle f, g\rangle = \int_0^{2\pi} \bar{f}(x)g(x)dx$. Let X_s be the periodic function

$$X_s(x) = \begin{cases} 1 \text{ if } |x - s - 2\pi n| < 1 \text{ for some integer } n \\ 0 \text{ otherwise} \end{cases}$$

and let $\mathcal{H}_0$ be the subspace of $\mathcal{H}$ consisting of finite linear combinations of the X_s's, i.e., functions of the form

$$f(x) = \sum_{k=1}^{N} a_k X_{s_k}(x).$$

Prove that the closure of $\mathcal{H}_0$, $\bar{\mathcal{H}}_0$, is $\mathcal{H}$; or equivalently, show that if $g \in \mathcal{H}$, and $\langle X_s, g\rangle = 0$ for all $s \in [-\pi, \pi]$, then $g = 0$. Hint: Consider the integral

$$\int_{-\pi}^{\pi} e^{-ins}\langle X_s, g\rangle ds.$$

REAL ANALYSIS GENERAL EXAM, AUGUST, 2003

In each problem, justify your assertions, show calculations, and identify those theorems which you invoke in your arguments.

1. (a) Show that for $x \geq 0$, $|\ln(1+x) - x| \leq x^2$.

(b) Show that

$$\sum_{n=1}^{\infty} \left(\ln(1 + \frac{1}{n^\alpha}) - \frac{1}{n^\alpha} \right)$$

is summable for $\alpha > 1/2$.

2. (a) Let A be a Lebesgue measurable subset of the real line R with $m(A) > 0$. Show that for any $\delta \in (0,1)$ there is a finite interval J such that $m(A \cap J) > (1-\delta)m(J)$.

(b) Give an example of a Lebesgue measurable subset A of R with $m(A) > 0$ and having the property that $m(A \cap J) < m(J)$ for all finite intervals J.

3. Let f and f' be continuous, real-valed functions on the interval $[a,b]$, and suppose $f(a) = f(b) = 0$ and $\int_a^b [f(x)]^2 dx = 1$. Prove

(a)

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and

(b)

$$\int_a^b [f'(x)]^2 dx \int_a^b [x f(x)]^2 dx \geq \frac{1}{4}.$$

4. For any x in the real line R, let I_x denote the open interval $(x-1, x+1)$. Suppose E is a Lebesgue measurable subset of R. Define a set F by

$$F = \{x \in R : m(I_x \cap E) > 0\}.$$

Prove that F is Lebesgue measurable.

5. Let f_n be a sequence of non-negative measurable functions on $[0, 1]$. If $f_n(x) \to f(x)$ almost everywhere (with respect to Lebesgue measure) and

$$\int_{[0,1]} f_n(x)dm \to \int_{[0,1]} f(x)dm$$

as $n \to \infty$, prove that for every measurable subset $E \subset [0, 1]$,

$$\lim_{n\to\infty} \int_E f_n(x)dm = \int_E f(x)dm.$$

6. Let $f : (0, \infty) \to R$ with $\lim_{x\to\infty} f(x) = L$. Define $f_n(x) = f(nx)$ for $0 < x < \infty$. Show that $f_n(x) \to L$ uniformly on $[a, \infty)$ for every $a > 0$, and that $f_n(x)$ converges uniformly to L on $(0, b)$ $(b > 0)$ if and only if $f(x) \equiv L$ on $(0, \infty)$.

GENERAL EXAM in REAL ANALYSIS January 2004

1. Suppose that $\{f_n\}$ is a sequence of continuous real-valued functions on $[0,1]$ with $f_1(x) \geq f_2(x) \geq f_3(x) \geq \ldots$, and

$$\lim_{n\to\infty} f_n(x) = 0$$

for all x. Show that $\{f_n\}$ converges uniformly.

2. Suppose that $\{g_n\}$ is an $L^2(\mathbf{R}, dx)$-sequence of real-valued functions such that

$$\sum_n \int g_n(x)^2\, dx < \infty.$$

   (a) Show that

$$\lim_{n\to\infty} g_n(x) = 0 \text{ a.e.}$$

   To do this it may be helpful to show that $\sum_n g_n(x)^2$ is finite a.e.

   (b) Determine

$$\lim_{n\to\infty} \int_{\mathbf{R}} \frac{\cos\left(g_n(x)\right)}{1+x^2} dx,$$

   and justify your answer.

3. Recall that $\{e^{2\pi i n x}\}$, $n \in \mathbf{Z}$, is an orthonormal set in $L^2([0,1], dx)$ with inner product

$$\langle f, g\rangle = \int_0^1 \bar{f}(x) g(x)\, dx.$$

   (a) Using geometric series, compute the Fourier series for

$$f_a(x) = \frac{1}{a - e^{2\pi i x}}$$

   for a real, with $a > 1$.

   (b) Compute, for a, $b > 1$,

$$\langle f_a, f_b\rangle.$$

(c) Show that if $g(x)$ is a trigonometric polynomial of the particular form

$$g(x) = \sum_{0 \le n \le N} c_n e^{2\pi i n x},$$

then

$$\lim_{a \downarrow 1} \langle f_a, g \rangle = g(0) = \sum_{0 \le n \le N} c_n.$$

4. Let f be a real-valued continuous function on $\mathbf{R}^d$, and let $\mathcal{A}_f$ be the collection of subsets of $\mathbf{R}^d$,

$$\mathcal{A}_f = \{A \,|\, A = f^{-1}(B) \text{ for some Borel set } B \subset \mathbf{R}\}.$$

(a) Show that $\mathcal{A}_f$ is a σ-algebra and a sub-algebra of the Borel σ-algebra on $\mathbf{R}^d$.

(b) Suppose that $g(x)$ is an $\mathcal{A}_f$ simple function on $\mathbf{R}^d$,

$$g(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$$

with χ_{A_i} the characteristic function for the $\mathcal{A}_f$-measurable set A_i.
Show that there exists a Borel measurable (simple) function h on $\mathbf{R}$ such that

$$g(x) = h(f(x)).$$

(c) Suppose that $f(x, y) = x^2 + y^2$ on $\mathbf{R}^2$. What do the sets of $\mathcal{A}_f$ look like, i.e. what symmetry property do they possess? What do $\mathcal{A}_f$-measurable functions depend on?

5. Suppose $1 < p < \infty$ and let $f \in L^p(\mathbf{R}, dx)$. Show that

$$\lim_{N \to \infty} \frac{1}{N^{(1-\frac{1}{p})}} \int_{-N}^{N} f(x)\, dx = 0.$$

ANALYSIS GENERAL EXAM, January 2005

In each problem justify all assertions, show calculations, and identify those theorems which you invoke in your arguments.

1. Let $\{f_n\}$ be a sequence of bounded measurable functions on $\mathbf{R}$.

   (a) Show from the definition of measurable functions that

$$g(x) \equiv \sup_n f_n(x)$$

   and

$$f(x) \equiv \limsup_n f_n(x)$$

   are measurable.

   (b) Assuming that the sum and difference of measurable functions are measurable and using your results of part (a), show that $E \equiv \{x : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

2. Let $\{f_n\}$ be a sequence of functions in $L^1([0,1], dx)$ with the properties that $f_n(x)$ converges pointwise a.e.to a function $f(x)$, and that the f_n's are uniformly integrable in the sense that for any $\epsilon > 0$, there exists an $M = M(\epsilon)$ such that

$$\int_{\{x:|f_n(x)|\geq M\}} |f_n(x)|\, dx < \epsilon$$

   (independent of n).

   (a) Show that

$$\int_{\{x:|f(x)|\geq M\}} |f(x)|\, dx \leq \epsilon.$$

   and that f is an L^1 function.

   (b) Show that the sequence $\{f_n\}$ converges to f in L^1.

3. Consider the infinite series,

$$F(x,t) = \sum_{n\geq 0} u_n(t) \cos(2\pi n x)$$

   $t, x \in [0,1]$ where each of the functions $u_n(t)$ is integrable, and

$$\sum_{n\geq 0} \int_0^1 |u_n(t)|\, dt < \infty.$$

   (a) Show that $G(x) \equiv \displaystyle\int_0^1 F(x,t)\, dt$ is continuous in x.

   (b) Use the Weierstrass M-test to show that $F(x,t)$ is continuous in x, for almost every $t \in [0,1]$.

4. Let $f(x)$, $g(x)$ be two normalized functions in $L^2(\mathbf{R}, dx)$, $\|f\|_2 = \|g\|_2 = 1$, and let

$$H(y) = \int f(y-x)g(x)dx.$$

(a) Prove that $H(y)$ satisfies the pointwise bound $|H(y)| \le 1$ and is continuous.

(b) Prove that $\lim_{y\to\infty} H(y) = 0$.

5. Show that if f is entire and $|f(z)| \le 1 + |z|^{1/2}$ for all z, then f is constant.

6. (a) Suppose that the meromorphic function g has a simple pole at z_0 and σ_r is the circular arc $\{z_0 + re^{i\theta}, \theta_0 \le \theta \le \theta_0 + \alpha\}$ for fixed θ_0 and $\alpha > 0$. Show that

$$\lim_{r\to 0^+} \int_{\sigma_r} g(z)dz = i\alpha\,\text{res}(g(z), z_0)$$

where $\text{res}(g(z), z_0)$ is the residue of g at z_0.

(b) Show that for a fixed c,

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{iz}}{z-c}\,dz = 0$$

where Γ_R is the semicircular arc in the upper-half of the complex plane,

$$\Gamma_R = \{z : |z| = R,\ 0 \le \theta \le \pi\}$$

(c) Calculate for any real number c,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x-c}\,dx \equiv \lim_{R\to\infty,\ \epsilon\to 0^+} \left(\int_{-R}^{c-\epsilon} \frac{\sin x}{x-c}\,dx + \int_{c+\epsilon}^{R} \frac{\sin x}{x-c}\,dx \right)$$

7. Suppose f, g are analytic on the unit disk $D = \{z : |z| < 1\}$, with g mapping D one-to-one and onto a region $G = g(D)$ contained in the complex plane. If $f(0) = g(0)$ and the range of f is contained in G, show that

(a) $|f'(0)| \le |g'(0)|$ and

(b) $f(rD) \subseteq g(rD)$ for all $0 < r < 1$.

8. Suppose that μ is a finite, positive measure in $\mathbb{C}$, supported on a compact set K. Set

$$f(z) = \int_{\mathbb{C}} \frac{d\mu(w)}{w-z}.$$

(a) Show that $f(z)$ is analytic in $\mathbb{C}\backslash K$ and that $\lim_{|z|\to\infty} f(z) = 0$.

(b) Show that for every disk Δ in $\mathbb{C}$, $\int_\Delta |f|dm < \infty$, where m is two-dimensional Lebesgue measure on $\mathbb{C} = \mathbf{R}^2$.

Complex Analysis, May 1977

1. Classify the singularities of the following functions, including $z = \infty$. Give orders of poles.

   (a) $\dfrac{z}{\sin z}$

   (b) $\frac{1}{(1+z^2)(2-z)^2}$

2. Consider
$$f(z) = \frac{1}{(1+z^2)(2-z)^2}.$$

   (a) Find its principal part at $z = 2$.

   (b) In what region does its Laurant series about $z = 2$ converge?

   (c) In what circular regions center at $z = 0$ does $f(z)$ have a Laurant expansion?

   (d) Expand $f(z)$ in $1 < |z| < 2$.

3. Integrate, by complex methods,

   (a) $\displaystyle\int_{-\pi}^{\pi} \frac{1}{1 - a\cos\theta}\, d\theta, \quad 0 < a < 1;$

   (b) $\displaystyle\int_{0}^{\infty} \frac{dx}{x^{\alpha}(x+1)^2}, \quad 0 < \alpha < 1.$

4. Find, if possible, a conformal map of A onto the upper half plane $\{z : \text{Im } z > 0\}$, where

   (a) A is the complex plane;

   (b) A is the unit disc $\{z : |z| < 1\}$.

Complex Analysis, November 1977

1. Identify and classify the singularities of the function $z\csc(z^2)$.

2. Let $f(z)$ be non-constant and meromorphic on $\mathbb{C}$. Assume that

$$f(z) = f(z+1) = f(z+i)$$

for all z in $\mathbb{C}$. Prove that $f(z)$ has at least one pole in the rectangle

$$\{z : Z = x + iy \text{ where } 0 \le x \le 1 \text{ and } 0 \le y \le 1\}.$$

3. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx$$

using the methods of complex analysis.

4. Assume that $p(z)$ is a polynomial such that $p(e^{i\theta})$ is real-valued for all real values of θ. Use the methods of complex analysis to show that $p(z)$ is identically equal to a constant.

5. Prove the theorem which asserts that if $\{f_n\}_1^\infty$ is a sequence of functions analytic in the disk $D = \{z : |z| < 1\}$ such that $|f_n(z)| > 0$ for all $n = 1, 2, 3, \ldots$ and all $z \in D$, and if $\lim_{n\to\infty} f_n = f$ uniformly on all compact subsets of D, then either $f(x) \equiv 0$ or $|f(z)| > 0$ for all $z \in D$.

Complex Analysis, May 1978

1. Classify all of the singularities of the function

$$f(z) = \frac{e^{1/z}}{1+z^2}$$

in the extended complex plane.

2. Determine the number of zeros of the polynomial

$$p(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$$

   (a) on the disk $\{z : |z| < 1\}$;
   (b) on the disk $\{z : |z| < 2\}$.

3. Prove or disprove:
   (a) There is a sequence of polynomials $\{p_n\}$ such that $\lim_{n\to\infty} p_n(z) = \frac{1}{z}$ uniformly on the circle $|z| = 1$.
   (b) There is a sequence of polynomials $\{p_n\}$ such that $\lim_{n\to\infty} p_n(z) = \frac{1}{z}$ uniformly on the circle $|z - 3| = 2$.

4. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^6}\, dx$ using the residue calculus.

5. Prove that the range of a non-constant entire function is a dense subset of the complex plane.

6. Let h be a measurable function on the positive real line with

$$\int_0^\infty |h(t)| dt < \infty.$$

For Im $z > 0$, set

$$H(z) = \int_0^\infty h(t) e^{itz} dt.$$

Verify in detail that $H(z)$ is a well-defined analytic function in Im $z > 0$.

Complex Analysis, September 1978

1. Compute by residue calculus

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$;

   (b) $\displaystyle\int_{0}^{\infty} \frac{x^{i\alpha}}{1+x^2}\, dx$, α real.

2. State and prove the maximum modulus theorem.

3. Let $f(z)$ be entire, with $f(z+2\pi i) = f(z)$. Prove that there exists a function $g(z)$ analytic on $\mathbf{C} - \{0\}$ such that

$$f(z) = g(e^z).$$

4. Find a one-one conformal mapping of the first quadrant $\{(x,y) : x > 0, y > 0\}$ onto the unit disc $\{(x,y) : x^2 + y^2 < 1\}$.

5. Classify the singularities of the following functions and find the principal parts at each singularity

   (a) $\displaystyle\frac{1}{z^2 + z^4}$

   (b) $\displaystyle\sin\left(\frac{1}{z}\right)$

6. Let $f(z)$ be entire and assume that

$$|f(z)| \leq A + B|z|^n$$

for some constants A and B. What can $f(z)$ be? Prove your answer.

Complex Analysis, May 1979

1. Prove that $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ converges and represents an analytic function in the half-plane Re $z > 1$.

2. Assume that $f(z)$ is analytic in the extended complex plane except for a finite number of poles. Prove that $f(z)$ is a rational function.

3. What is $\sup\limits_{f \in S} |f'(0)|$ where S is the set of functions which are analytic and bounded by 1 in $|z| < 1$. Justify your answer.

4. Evaluate
$$I_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-in\theta}}{z + \cos\theta}\, d\theta$$
for any integer n, positive, negative, or zero.

5. Assume that $f(z)$ is analytic in a region containing the closed strip
$$S = \{z = x + iy : 0 \leq x \leq 1, -\infty < y < \infty\}.$$
Assume that
$$\lim_{|y| \to \infty} \left(\sup_{0 \leq x \leq 1} |f(x + iy)| \right) = 0.$$
Suppose that for all real y,
$$|f(iy)| \leq a \text{ and } |f(1 + iy)| \leq b,$$
where $a > 0$ and $b > 0$. Prove that
$$|f(x + iy)| \leq a^{1-x} b^x$$
in all of S. (Hint: Show that the function $g(z) = \frac{f(z)}{a^{1-z} b^z}$ is bounded by 1 in S, and deduce the inequality from this.)

Complex Analysis, September 1979

1. Prove the Fundamental Theorem of Algebra by complex methods.

2. Prove that $\sqrt{1 - z^2}$ has a single-valued branch in the complex plane, cut along the interval $-1 \leq x \leq 1$.

3. Find the Laurent expansion of the function

$$f(z) = \frac{1}{(1 + z^2)(9 - z^2)}$$

in an annulus $r_1 < |z| < r_2$ containing the point $z = 2$. Find r_1 and r_2 for the largest such annulus.

4. Evaluate the following by complex methods

$$\int_0^\infty \frac{\cos ax}{1 + x^2}\, dx, \quad a > 0.$$

5. Prove that the mapping defined by an analytic function $f(z)$ is conformal at all points where $f'(z)$ does not vanish. ("conformal" means "angle-preserving").

Complex Analysis, April 1980

1. Consider
$$f(z) = \frac{1}{(1+z^2)(2-z)^2}.$$
   (a) Find its principal part at $z = 2$.
   (b) In what region does its Laurent series about $z = 2$ converge?
   (c) In what circular regions centered at $z = 0$ does $f(z)$ have a Laurent expansion?
   (d) Expand $f(z)$ in $1 < |z| < 2$.

2. Let $f(z)$ be entire and assume that
$$|f(z)| \leq A + B|z|^n$$
for some constants A and B. What can $f(z)$ be? Prove your answer.

3. Compute by residue calculus
$$\int_0^\infty \frac{x^{-i\alpha}}{1+z^2}\,dx, \qquad \alpha \text{ real}.$$

4. Prove that $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ converges and represents an analytic function in the half-plane Re $z > 1$.

5. Find, if possible, a conformal map of A onto the upper half plane $\{z : \text{ Im } z > 0\}$ where
   (a) A is the complex plane;
   (b) A is the unit disk $\{z : |z| < 1\}$.

Complex Analysis, September 1980

1. Find the image of

   (a) the unit disc $|z| < 1$, and

   (b) the line segment $[-i, i]$ under the map
   $$w = \frac{z-1}{z+1}.$$

2. Find the possible values of i^i.

3. Classify singularities of
   $$\frac{\sqrt{z}\sinh\sqrt{z}}{\sin^2 z}.$$

4. Does $\log(1-z^2)$ have a single-valued branch on the region $\mathbf{C} - [-1, +1]$? Justify your answer.

5. Evaluate
   $$\int_0^\infty \frac{\sin x}{x}\, dx$$
   by complex methods.

6. Is there a function $f(z)$, analytic on a neighborhood of $|z| \leq 1$ which maps the unit circle $|z| = 1$ with positive orientation into the unit circle with negative orientation? Give an example or a disproof.

Complex Analysis, January 1981

1. Let $f(z)$ be analytic in a domain Ω. Prove that $u(x,y) = \operatorname{Re} f(x+iy)$ is a harmonic function on Ω.

2. Find and classify the singularities of the following functions in the complex plane:

   (a) $\dfrac{e^{1/z}}{z(1+z)^2}$

   (b) $\cos\sqrt{z}$

   (c) $\dfrac{\log(1+z^2)}{1-z^4}$

3. Evaluate by contour integration
$$\int_0^\infty \frac{\cos ax}{1+x^2}\,dx.$$

4. Let C be the positively oriented unit circle $|z|=1$. Find
$$\int_C \frac{dz}{z^2 \sin z}.$$

5. Define
$$(*) \qquad f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n,$$
where
$$a_n = \begin{cases} 2^{-(n+1)} & n \ge 0; \\ 1 & n < 0, \end{cases}$$
for those values of z for which the series converges.

   Let $f(z)$ be defined elsewhere by analytic continuation.

   (a) Where does the representation $(*)$ hold?

   (b) What sort of behavior does $f(z)$ have at the origin $z=0$?

6. Prove that there is a positive number R for which there exist complex numbers $c_0, c_1, c_2, \ldots$ such that
$$\frac{(z^2+\pi^2)(e^z-1)}{\sinh z} = \sum_{n=0}^{\infty} c_n z^n$$
for $|z|<R$. Is there a largest such R? If so, what is it? Justify your answers.

Complex Analysis and Linear Spaces, September 1988

1. Construct a conformal mapping of the unit disk $|z| < 1$ onto the strip $0 < \text{Re } z < 1$.

2. Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on an open subset Ω of the complex plane which is uniformly bounded on every compact subset of Ω. Suppose that

$$\lim_{n\to\infty} f_n(z)$$

exists for every z in Ω. Use Cauchy's integral formula to show that the convergence is uniform on compact subsets of Ω.

3. Let S be the set of complex numbers $z = x + iy$ with x and y non-negative. Let $f(z)$ be holomorphic in a neighborhood of S and satisfy $f(z) \to 0$ and $|z| \to \infty$ inside S. Prove that

$$\lim_{b\to\infty} \int_0^b f(t)e^{it}dt = \int_0^{\infty} f(it)e^{-t}dt.$$

4. Define

$$\lambda(f) = \int_0^1 x^{-2/3} f(x)dx$$

for every continuous function $f(x)$ on $[0, 1]$. Show that there is no continuous linear functional μ on $L^3(0, 1)$ such that

$$\mu(f) = \lambda(f)$$

for all continuous functions $f(x)$ on $[0, 1]$.

5. Prove or disprove: $C([0, 1])$ is a Hilbert space in the inner product defined by

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$$

for all elements f and g of the space.

6. Prove that there is a positive number R for which there exist complex numbers $c_0, c_1, c_2, \ldots$ such that

$$\frac{(z^2 + \pi^2)(e^z - 1)}{\sinh z} = \sum_{n=0}^{\infty} c_n z^n$$

for $|z| < R$. Is there a largest such R? If so, what is it? Justify your answers.

Complex Analysis and Linear Spaces, January 1989

1. (a) Let f be holomorphic on a region G in the complex plane. If $u = \text{Re } f$ and $v = \text{Im } f$, show that u and v are harmonic on G.

   (b) Exhibit real-valued harmonic functions u, v in the complex plane such that $f = u+iv$ is not holomorphic in the plane (prove this).

2. (a) Compute the power series expansion of $\sinh z$ about the point $z_0 = \pi i$.

   (b) Evaluate
$$\lim_{z\to\pi i} \frac{\sinh z}{z - \pi i}.$$

3. Let $f(z)$ be holomorphic in the unit disk with
$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$
Assume that
$$|f(z)| \le 1/(1 - |z|)$$
for all z in the disk. Prove that for $n \ge 1$,
$$|a_n| \le (n+1)\left(1 + \frac{1}{n}\right)^n.$$

4. Assume that
$$\lim_{n\to\infty} \int_0^1 \left| f_0(x) - \sum_{k=0}^{n} a_k x^k \right|^2 dx = 0,$$
where $a_0, a_1, a_2, \ldots$ are complex numbers and $f_0(x)$ is a function in $L^2(0,1)$. Show that

   (a) $\lim_{n\to\infty} \int_0^1 \sum_{k=0}^{n} a_k x^k \, dx$ exists as a finite number;

   (b) $\lim_{k\to\infty} \frac{a_k}{k+1} = 0$;

   (c) the power series
$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$
   converges in the disk $|z| < 1$;

   (d) $f(x) = f_0(x)$ a.e. on $(0,1)$.

5. Let $f, g \in L^2(-\infty, \infty)$.

(a) Prove that

$$\lim_{h \to 0} \int_{-\infty}^{\infty} |f(x+h) - f(x)|^2 dx = 0$$

for all real x. (Hint: First prove this under the additional assumption that $f(x)$ is continuous and has compact support.)

(b) Prove that

$$F(x) = \int_{-\infty}^{\infty} f(x+t)g(t)dt$$

is a continuous function of real x.

Complex Analysis and Linear Spaces, September 1990

1. Show that the integral
$$F(x) = \int_{-\infty}^{\infty} e^{-t^4} e^{zt} dt$$
converges absolutely for all complex numbers z and that $F(z)$ is an entire function.

2. Let f be holomorphic in a region Ω which contains the closed unit disk $\bar{D} = \{z : |z| \leq 1\}$. If $|f(z)| < 1$ for $z \in \bar{D}$, how many roots does the equation $f(z) = z$ have in $\bar{D}$? (Hint: Use Rouche's Theorem.)

3. Suppose that $f(z)$ is an entire function satisfying
$$|f(z)| \leq A + B|z|^k$$
for all complex z. Prove that $f(z)$ is a polynomial.

4. Set $\beta = (\pi/2)^{\frac{1}{2}}$. Show that for $x > 0$,
$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{ixt}}{\cosh(\beta t)}\, dt = \frac{1}{\cosh(\beta x)}.$$
Justify each step. (Hint: Use the residue theorem with an expanding sequence of rectangular contours with vertices $\pm R_N + i0$ and $\pm R_n + iR_n$ with R_n chosen so that the contours avoid the zero of $\cosh(\beta z)$.)

5. (a) Suppose that x and y are vectors in a Hilbert space. Prove that
$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$
   (b) Explain why the identity in (a) is called the parallelogram law.
   (c) Prove that
$$\|y\|^2 \left[\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2\right] = \left\| \|y\|^2 x - \langle x, y\rangle y \right\|^2.$$
   (d) State the Schwarz inequality for vectors x and y and derive it from (c).
   (e) Suppose equality holds in the Schwarz inequality. What can you say about the vectors x and y? Prove it.

Complex and Linear Analysis, May 1991

1. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+b^2}\,dx,$$
where $b > 0$.

2. (a) Suppose that f is analytic in an open set containing $\{z : |z| \le 1\}$, that $f(z)$ is never zero for $|z| < 1$ and that $|f(z)| = 1$ when $|z| = 1$. Show that f is constant.

   (b) For fixed z_0 with $|z_0| < 1$, let
$$h(z) = \frac{z - z_0}{1 - \overline{z}_0 z}.$$
Show that $|h(z)| < 1$ for $|z| < 1$ and that $|h(z)| = 1$ when $|z| = 1$.

   (c) Suppose that f is analytic in an open set containing $\{z : |z| \le 1\}$ and that $|f(z)| = 1$ whenever $|z| = 1$. Show that f is a rational function.

3. Let f be a one-to-one analytic function defined on the unit disk $D = \{z : |z| < 1\}$. Show that the complement of $f(D)$ in the plane contains an infinite number of points. (Hint: f^{-1} exists.)

4. (a) Suppose f is analytic on a region containing both a smooth simple closed curve C and its interior D. For a complex number w not lying on $f(C)$, evaluate
$$\int_C \frac{f'(z)z}{f(z) - w}\,dz.$$
Consider the cases $w \in f(D)$ and $w \notin f(D)$.

   (b) Suppose C is the circle $\{z : |z| = 2\}$. Use (a) to evaluate
$$\int_C \frac{e^z z}{e^z - i}\,dz.$$

5. (a) Let U be an open set which intersects the real line $\mathbf{R}$. Let h be continuous on U and analytic on $U \setminus \mathbf{R}$. Prove that h is analytic on U.

   (b) Let f be an entire function such that $f(z + 2\pi) = f(z)$ for all z. Prove that there exists g analytic on $\mathbf{C} \setminus \{0\}$ such that
$$f(z) = g(e^{iz}).$$

   (c) For f as in (b) prove that
$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{inz}$$
for some coefficients a_n.

(d) For a_n as in (c) prove that

$$|a_n|A^n \to 0 \text{ as } |n| \to \infty, \quad \forall\, A > 0.$$

6. Evaluate

$$\sup_f \left| \int_0^1 \frac{f(x)}{\sqrt{x}}\, dx \right|,$$

where the supremum is taken over all continuous functions f on $[0,1]$ satisfying

$$\int_0^1 |f(x)|^4 dx \le 1.$$

General Exam: Complex and Linear Analysis
January 1992

Name:

If a major theorem is used, it should be identified by name or statement; it is essential to verify that the hypotheses of such theorems are satisfied.

1. Let $f(z) = \frac{1}{z}$, a function which is analytic on the annular region
$$G = \{z : \frac{1}{2} < |z| < \frac{3}{2}\}.$$
Prove or disprove: There exist polynomials p_n, $n = 1, 2, 3, \ldots$, such that
$$\lim_{n \to \infty} p_n(z) = f(z)$$
uniformly on every compact subset of G.

2. Let a be any real number. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{e^{iax}}{e^x + e^{-x}} dx$$
by integrating a certain analytic function around the rectangle with corners at $-r, r, r + \pi i, -r + \pi i$, and letting $r \to \infty$. Perform all necessary estimates.

3. Show that the polynomial
$$2z^5 + 8z - 1$$
has exactly 4 zeros (counting multiplicities) in the annulus $\{z : 1 < |z| < 2\}$.

4. Let a and b be distinct complex numbers.

   (i) Find a convergent expansion of the form
$$\frac{1}{z-a} = \sum_{n=0}^{\infty} c_n (z-b)^n.$$
   In what region is the expansion valid?

(ii) Find a convergent expansion of the form

$$\frac{1}{z-a} = \sum_{n=0}^{\infty} d_n \frac{1}{(z-b)^n}.$$

In what region is this expansion valid?

5. Let U denote the open unit disk $\{z : |z| < 1\}$. The Dirichlet integral of a function f analytic on U is the (possibly infinite) integral

$$D(f) \stackrel{\text{def}}{=} \int\int_U |f'(x+iy)|^2 dx\, dy.$$

(a) Suppose that f is analytic on U with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that $D(f) < \infty$ if and only if $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$.

(b) Suppose that f is analytic and one-to-one on U. What is the geometric interpretation of $D(f)$?

(c) Show that there are unbounded functions f analytic on U with $D(f) < \infty$. (Hint: Part (b) and the Riemann Mapping Theorem.)

6. Let $f \in L^2(-\infty, \infty)$ and consider the Fourier transform $\hat{f}$, defined by

$$\hat{f}(y) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-ixy} f(x) dx.$$

The limit here is *limit in mean*, that is, in the $L^2(-\infty, \infty)$-norm. Suppose also that $b > 0$ and that $\hat{f}(y) = 0$ a.e. outside of $[-b, b]$. Thus you can consider $\hat{f}$ as an element of $L^2(-b, b)$, and so it has a Fourier *series*

$$\hat{f}(y) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{\pi i n y}{b}}, -b \le y \le b.$$

(a) Calculate the coefficients $\{c_n\}_{n=-\infty}^{\infty}$ as integrals, and recognize these integrals in terms of the values of f on a certain arithmetic progression.

(b) Evaluate the sum

$$\sum_{n=-\infty}^{\infty} |f(\frac{\pi n}{b})|^2$$

in terms of $\int_{-\infty}^{\infty} |f(x)|^2 dx$.

General Exam: Analysis (Part II)
September 12, 1992

Name:

1. Find the Laurent series expansion for

$$f(z) = \frac{1}{(z-2)(z-3)}$$

for

   (a) $2 < |z| < 3$, and
   (b) $3 < |z| < \infty$.

2. Let $u(x, y) = e^{ax} \cos by$, where a and b are real numbers. For which pairs a, b does there exist a function $f(z)$, where $z = x + iy$, with $u = Re\, f$. In these cases, find f.

3. Evaluate $\int_0^\infty \frac{1}{(1+x)\sqrt{x}}\, dx$ by residues.

4. (a) Let f be analytic and satisfy $Re\, f(z) \geq 0$ in the entire complex plane. Prove that f is constant.
   (b) Construct a non-constant analytic function f on the slit plane $\mathbf{C} \setminus (-\infty, 0]$ satisfying $Re\, f(z) \geq 0$ for all z in the slit plane.
   (c) Prove or give a counterexample to this statement: If f is analytic and satisfies $Re\, f(z) \geq 0$ on the punctured plane $\mathbf{C} \setminus \{0\}$, then f is constant.

5. If $f \in L^1(\mathbf{R})$, the Fourier transform of f is the function

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-ixt} f(t) dt.$$

(a) Find $\hat{f}$ if

$$f(t) = \left\{ \begin{array}{ll} 1 & \text{if} \quad |t| \leq a \\ 0 & \text{if} \quad |t| > a \end{array} \right\}$$

where $a > 0$.

(b) Evaluate the integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin\, ax}{x} \right) dx, \quad a > 0.$$

(c) Show that there exists $f \in L^1(\mathbf{R})$ with

$$\hat{f}(x) = e^{-|x|} \frac{\sin\, x}{x},$$

and find $f(0)$.

6. Let $A = \{z \in \mathbf{C} : 1 < |z| < 2\}$. Let G be the set of all functions f analytic on A which have no zeros in A. Observe that G is a group under pointwise multiplication. Fix a circle γ of radius r and center at 0, oriented counterclockwise, where $1 < r < 2$, and define $\Phi : G \to \mathbf{C}$ by

$$\Phi(f) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)}\, dz.$$

(a) Show that Φ is independent of r.

(b) Show that Φ is a group homomorphism from G to the additive group $\mathbf{C}$, that is, $\Phi(fg) = \Phi(f) + \Phi(g)$.

(c) Find the range of Φ.

(d) True or False, and prove your answer:

$$\text{Ker}\ \Phi = \{e^g : g \text{ is analytic on } A\}.$$

Complex Analysis General Exam
January 1993

Name:

1. Let $f(x) = z \sin \frac{1}{z}$ for all nonzero complex numbers z.

   (a) Locate all zeros of $f(z)$ and determine their orders.

   (b) Examine the isolated singularities of $f(z)$ at 0 and ∞. Determine their nature, that is, are they poles, essential, or removable? If removable, say how to define the function at that point to make it analytic.

2. Find all entire functions $f(z)$ such that the inequality

$$|f(z)| \leq c(1 + |z|^2)$$

   is satisfied for all complex z and some constant c.

3. For what complex-valued measurable functions $g(x)$ on $[0, \infty)$ does the inequality

$$|\int_0^\infty g(x)u(x)dx|^2 \leq \int_0^\infty x^2 |u(x)|^2 dx$$

   hold for all continuous, complex-valued functions $u(x)$ with compact support on $(0, \infty)$.

4. For $t \in R$ define the fractional linear transformation

$$f_t(z) = \frac{z + \tanh \frac{\pi t}{4}}{1 + z \tanh \frac{\pi t}{4}}, \quad z \in C.$$

   (i) Show that f_t maps the unit disk $|z| < 1$ onto itself.

   (ii) Show that if $t_1, t_2 \in R$, then

$$f_{t_1}(f_{t_2}(z)) = f_{t_1 + t_2}(z).$$

   (iii) Show that

$$g(z) = \frac{2}{\pi} \log \frac{1 + z}{1 - z}$$

   maps the disk $|z| < 1$ one-to-one onto the strip $|\text{Im } w| < 1$, and that

$$g(f_t(z)) = g(z) + t, \qquad |z| < 1, \ t \in R.$$

5. Show that an analytic branch of

$$f(z) = \log \frac{z+1}{z-1}$$

can be defined on $|z| > 1$, and find its Laurent expansion.

6. Evaluate

$$\int_0^\infty \frac{\log\, x}{1+x^2}\, dx$$

by the method of contour integration.

7. Let k and m be integers, $k \geq 2$, and $0 \leq m \leq k-1$. Show that there exist constants $a_0, \ldots, a_{k-1}$ such that

$$\frac{z^m}{z^k - 1} = \sum_{j=0}^{m-1} \frac{a_j}{z - e^{2\pi i j/k}}.$$

Then find formulas for these constants.

Analysis General Exam, Pt. 2

September 3, 1994

Justify your answers with as much detail as you can.

1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions analytic on an open set G in the complex numbers $\mathbb{C}$. Suppose there exists $M > 0$ such that $|f_n(z)| \le M$ for all z in G and all n. Suppose further that for each z in G, $\{f_n(z)\}_{n=1}^{\infty}$ is a convergent sequence of complex numbers. Show that the sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ converges uniformly on compact subsets of G.

2. Let $\mathbb{D} = \{z : |z| < 1\}$, and let $\mathcal{F}$ denote the set of analytic functions mapping $\mathbb{D}$ to itself.

   (a) Let
   $$\lambda = \sup\left\{\left|f\left(\frac{1}{2}\right)\right| : f \in \mathcal{F} \text{ and } f(0) = 0\right\}.$$
   Find λ and determine all g in $\mathcal{F}$ with $g(0) = 0$ and $\left|g\left(\frac{1}{2}\right)\right| = \lambda$.

   (b) Let
   $$\mu = \sup\left\{\left|f'\left(\frac{1}{2}\right)\right| : f \in \mathcal{F} \text{ and } f\left(\frac{1}{2}\right) = 0\right\}.$$
   Find μ, and determine all h in $\mathcal{F}$ with $h\left(\frac{1}{2}\right) = 0$ and $\left|h'\left(\frac{1}{2}\right)\right| = \mu$.

3. Find all possible values of
   $$\int_{\gamma} z^3 \cos\frac{1}{z} dz,$$
   where γ is a piecewise C^1 closed curve not passing through 0.

4. Use the method of residues to evaluate
   $$\int_0^{\infty} \frac{\sqrt{x}}{(1+x)^2}\, dx.$$

5. Thoroughly discuss the mapping properties of
   $$f(x) = z + \frac{1}{z}.$$
   In particular, how does the argument principle, applied on the unit circle $\{z : |z| = 1\}$, give information about the range of f on the punctured disk $\{z : 0 < |z| < 1\}$?

6. Suppose f is analytic on $B = \{z : 0 < |z| < 1\}$ with an essential singularity at 0. Show that $f(B)$ is dense in the plane. (A proof is required, not merely quoting a theorem.)

Analysis General Exam, Part II

January 1996

Justify your answers with as much detail as you can.

1. (a) Cite a theorem which assures that $|e^{-z^2}|$ attains a maximum value on $\{z : |z| \le 1\}$.

   (b) Find the maximum value and *all* points where it is attained.

2. Suppose that f has a simple pole at $z = z_0$. Let γ_r be the circular arc $\{z_0 + re^{i\theta} : 0 \le \theta \le \alpha\}$, where $0 < \alpha < 2\pi$. Calculate

$$\lim_{r \to 0} \int_{\gamma_r} f(z) dz$$

   in terms of the residue of f at z_0.

3. How many zeros, if any (counting multiplicities), does $f(z) = z^3 + e^z$ have in $\{z : |z| < e\}$?

4. Let the polynomial

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n,$$

   with $a_n \ne 0$, have zeros $\zeta_1, \zeta_2, \ldots, \zeta_n$. By using a suitable contour C and suitable integers m, integrate $z^m p'(z)/p(z)$ over C to prove the formulas

$$\sum_{k=1}^{n} \frac{1}{\zeta_k} = -\frac{p'(0)}{p(0)},$$

   and

$$\sum_{k=1}^{n} \frac{1}{\zeta_k^2} = \left(\frac{p'(0)}{p(0)}\right)^2 - \frac{p''(0)}{p(0)}.$$

5. Suppose ϕ is a one-to-one analytic map of the unit disk $D = \{z : |z| < 1\}$ onto the simply connected domain Ω, and let $z_0 = \phi(0)$. Show that $|\phi'(0)| \geq r$ where r is the distance from z_0 to the boundary of Ω. What can you say about Ω if $|\phi'(0)| = r$? (Suggestion: Consider the map $\psi : D \to D$ given by $\psi(z) = \phi^{-1}(z_0 + rz)$.)

6. Calculate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$$

by the method of residues.

7. Consider the function

$$f(z) = \left(\frac{1+z}{1-z}\right)^2.$$

Is f one-to-one on the unit disk $D = \{z : |z| < 1\}$? What is the image of D under f?

General Exam in Analysis, Part II, Fall 1997

Please present your solutions as proofs, including all logical steps and detailed calculations. Verify or give adequate reasons for assertions that you make. Cite by name Theorems you invoke.

Please write only on one side of your paper.

Do problems 1–6 and either 7 or 8.

1. Find all expansions of $\frac{1}{(z-2)(z-3)}$ in integer powers of z (both positive and negative powers are allowed), and state where each converges.

2. Find an explicit conformal map of the strip $S = \{x + iy : \ 0 < x < 1\}$ onto the unit disk $\mathbb{D} = \{z : \ |z| < 1\}$.

3. Suppose that f is an analytic mapping from $\mathbb{D} = \{z : \ |z| < 1\}$ into itself. Suppose $a \in \mathbb{D}$ and $f(a) = 0$. Show that $|f(0)| \le |a|$.

4. (a) Calculate
$$\int_\Gamma \bar{z}\, dz,$$
where Γ is the straight line segment from 1 to $2 + 2i$.

   (b) Evaluate by residues:
$$\int_0^\infty \frac{1}{1+x^3}\, dx$$
(Your answer should be a *real* expression with no $\sqrt{-1}$ appearing.)

5. Show that $z^4 + z^3 + 1$ has exactly two zeros with positive real part.

6. Let $\Omega = \{z : \ 0 < |z| < 1\}$. Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence of distinct points in Ω with $a_n \to 0$ as $n \to \infty$. Suppose that f is analytic in Ω except for poles at each a_n (so 0 is not an *isolated* singularity of f). Prove: Given any $w \in \mathbb{C}$, there exists a sequence $\{z_n\}_{n=1}^\infty$ in Ω with $z_n \to 0$ and $f(z_n) \to w$.

7. Suppose that f_n is analytic in the open unit disk $\mathbb{D}$ and $|f_n(z)| \le M < \infty$ for z in $\mathbb{D}$ and $n = 1, 2, 3, \ldots$ Assume that for each z in $\mathbb{D}$, $f_n(z) \to 0$ as $n \to \infty$. Show that $f_n'(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$.

8. Let K be a compact subset of the interval $[-1/2, 1/2]$, and suppose the one-dimensional Lebesgue measure of K is zero. Suppose $f(z)$ is analytic and bounded in $\mathbb{D} \setminus K$, where $\mathbb{D} = \{z : |z| < 1\}$. Show that $f(z)$ extends to an analytic function on $\mathbb{D}$. (Hint: For some r, $\frac{1}{2} < r < 1$, let C be the circle $\{z : |z| = r\}$, and let

$$g(z) = \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Show that $f(z) = g(z)$ in $\{z \in \mathbb{D} \setminus K : |z| < r\}$.)

General Exam: Complex Analysis, Fall 1998

1. Suppose f is analytic in a domain $\Omega \supset \{z : |z| \leq 1\}$, $f(0) = 1$, and $|f(z)| > 2$ if $|z| = 1$. Must f have a zero in the unit disk? Prove or give a counterexample.

2. If $a > 1$ show that $z + e^{-z} = a$ has one and only one solution with positive real part.

3. The function
$$\frac{1}{1-z^2} + \frac{1}{z-2}$$
is to be expanded in a series of the form $\sum_{n=-\infty}^{\infty} c_n z^n$. How many such expansions are there? In what region is each valid? Find the coefficients c_n for each.

4. Let $f, f_1, f_2, \ldots$ be holomorphic functions which are defined on the unit disk $\mathbf{D} = \{z : |z| < 1\}$. Assume that
$$\lim_{n\to\infty} f_n(z) = f(z)$$
uniformly on all compact subsets of $\mathbf{D}$. Prove that
$$\lim_{n\to\infty} f_n'(z) = f'(z)$$
uniformly on all compact subsets of $\mathbf{D}$.

5. If f is an analytic function on the unit disk $\mathbf{D} = \{z : |z| < 1\}$ which maps $\mathbf{D}$ one-to-one and onto a domain $\Omega \supset \mathbf{D}$ with $f(0) = 0$, show $|f'(0)| \geq 1$. When does $|f'(0)| = 1$? Hint: If ϕ is the inverse function, apply the maximum principle to $\phi(z)/z$.

6. Compute the integral
$$\int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t}\, dt$$
in closed form for $0 < a < 1$. Hint: the rectangle $[R, R+2i\pi, -R+2i\pi, -R]$ might be useful.

General Examination **Complex Analysis** **August 1999**

Work all problems. Give your arguments in adequate detail.

1. Let a be any complex number. Show that the functions defined by the series

$$1 + az + a^2z^2 + \cdots$$

and

$$\frac{1}{1-z} - \frac{(1-a)z}{(1-z)^2} + \frac{(1-a)^2z^3}{(1-z)^2} - \cdots$$

are analytic continuations of each other. Describe the domains of validity (convergence) of the series.

2. Determine the radii of convergence of the power series which represents the following functions near $z = 0$:

   (a) $]fracze^z - 1$

   (b) $\dfrac{e^z - 1}{z^2 + 4\pi^2}$

3. Let the set $S = \{z \in \mathbf{C} : z \neq 1, 2, 3, \ldots\}$. Show that the series

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-z)^2}$$

converges uniformly and absolutely on every compact subset of S.

4. Evaluate by residues

$$\int_{-\infty}^{\infty} \frac{x+3}{x^3+27} dx.$$

5. Let $\{f_n\}$ be a sequence of entire functions with real zeros only and let $f = \lim_{n\to\infty} f_n$ exist uniformly on every compact set of $\mathbf{C}$. Prove that f has real zeros only.

General Examination Complex Analysis January 17, 2000

1. The function $\frac{1}{1-z^2} + \frac{1}{5-z}$ is to be expanded in a series of the form $\sum_{-\infty}^{\infty} c_n z^n$. How many such expansions are there? In what regions are they valid? Find the coefficients c_n for one of these regions.

2. Evaluate
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1}\, dx \text{ for } 0 < a < 1,$$
using a rectangular contour.

3. Suppose $f(z)$ is entire and real-valued on the segment $(-\epsilon, \epsilon)$ of the real axis for some $\epsilon > 0$. Show $f(x)$ is real for all real x. How are $f(z)$ and $f(\bar{z})$ related for $z \in \mathbb{C}$?

4. Show that if f is analytic in $\Omega \supseteq \bar{\mathbb{D}}$ where $\bar{\mathbb{D}}$ is the closed unit disk, then
$$\max_{|z|=1} \left| \frac{1}{z} - f(z) \right| \geq 1.$$

5. Determine the number of roots of $2z^5 + 8z - 1 = 0$ in
   (a) $|z| \geq 2$
   (b) $|z| < 1$
   (c) $1 < |z| < 2$

6. Suppose f is analytic in $\mathbb{D} = \{|z| < 1\}$ with $f(0) = 0$ and $\text{Re } f(z) < A$ for some $A > 0$. Show
$$|f(z)| \leq \frac{2A|z|}{1-|z|} \text{ for } z \text{ in } \mathbb{D}.$$
(Hint: consider $\frac{f(z)}{2A-f(z)}$.)

7. (a) Suppose f is a non-constant analytic function in an open set Ω containing the closed unit disk $\bar{\mathbb{D}}$ such that $|f|$ is constant on $\partial\mathbb{D}$. Prove f has at least one zero in $\mathbb{D}$.
   (b) Find all entire functions g with $|g(z)| = 1$ whenever $|z| = 1$.

General Examination **Complex Analysis** **August 2000**

In each problem, justify all assertions, show calculations and identify those theorems which you invoke in your arguments.

1. (a) Is the function $u(x,y) = x^3 - 3xy^2 + 3x^2y - y^3$ harmonic in the plane?
   (b) Does there exist a real function $v(x,y)$ so that $f = u + iv$ is analytic in the plane? Explain.
   (c) If yes in (b), find all such v.

2. Use the method of residues to calculate

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\, dx, \quad a > 0.$$

3. Show that the series

$$\frac{1}{1-z} + \frac{z}{z^2-1} + \frac{z^2}{z^4-1} + \frac{z^4}{z^8-1} + \cdots$$

converges to 1 for $|z| < 1$ and to 0 for $|z| > 1$.

4. (a) Classify all singularities of

$$f(z) = \frac{(e^{2iz}-1)^2}{z^5(z^2-\pi^2)^2}$$

   in the complex plane.
   (b) What is the nature of the singularity at $z = \infty$?

5. Let M be a fixed positive constant. Find all entire functions f such that

$$\left| \int_{|z-w|=1} \frac{f(z)}{(z-w)^3}\, dz \right| \leq M,$$

for all complex numbers w.

6. Let G be a bounded open connected set in the plane and assume that f is non-constant and continuous on $\bar{G}$, analytic in G, and that $|f(z)| = 1$ for all z in the boundary ∂G. Show that f has at least one zero in G.

7. Find all functions f analytic on $\{z : |z| < 2\}$ for which $\left|f\left(\frac{1}{n}\right)\right| \leq \frac{1}{3^n}$, $n = 1, 2, 3, \ldots$. Prove your answer.

1. Let $f(z)$ be the meromorphic function defined in the complex plane $\mathbf{C}$,

$$f(z) = z^{-2} \cot z = \frac{\cos z}{z^2 \sin z}.$$

Classify the singularities of $f(z)$ in $\mathbf{C}$, and, at each singularity, compute the associated residue there.

2. Compute the integrals

(a)

$$\oint_{|z|=1} \bar{z} dz$$

by parametrizing z. $\bar{z}$ is the complex conjugate of z.

(b)

$$\oint_{|z|=1} z^n e^{1/z} \, dz,$$

for all integers n.

(c)

$$\int_0^\infty \frac{x^\alpha}{1+x} dx$$

for $-1 < \alpha < 0$ using contour integral techniques.

(d)

$$\hat{f}(k) = \int_{\mathbf{R}} \frac{e^{ikx}}{(1+x^2)} dx$$

using contour integral techniques.

3. Suppose that $f(z)$ is analytic in a neighborhood of the closed unit disc $\bar{D}$ with $D = \{z : |z| < 1\}$ and satisfies the bound,

$$|f(z)| < 2$$

in D.

(a) Show that $f'(z)$ satisfies a bound

$$|f'(z)| \leq \frac{2}{1 - |z|}$$

for $z \in D$.

Now assume that $f(z)$ also satisfies the bound

$$|z|^n \leq |f(z)|$$

for $z \in \bar{D}$, with n a fixed non-negative integer.

(b) Give an upper bound on the winding number for f about 0 by estimating an appropriate integral with path of integration the circle $\{z : |z| = 1/2\}$. (You need not optimize your bound by varying the radius of the circle.)

4. Suppose that $f(z)$ is analytic in a neighborhood of the closed right-half plane $\bar{\mathbf{C}}_+$, $\mathbf{C}_+ = \{z \in \mathbf{C} : x > 0\}$, and satisfies the bounds,

$$|f(z)| \leq c(1 + |z|)$$

for $z \in \mathbf{C}_+$, c a fixed constant; and

$$|f(iy)| \leq (1 + y^2)^{1/2}$$

for z on the imaginary axis.

Show that $f(z)$ satisfies the sharp bound

$$|f(z)| \leq |1 + z|$$

for $z \in \bar{\mathbf{C}}_+$, i.e. $c = 1$. Hint: You might want to consider $f(z)/(1 + z)$ or possibly $f(z)/(1 + z)^{1+\epsilon}$.

5. (a) Let $D = \{z : |z| < 1\}$ be the unit disc in the complex plane. Give an example of a fractional linear function $f(z) = (z - z_0)/(cz + d)$ with a simple zero at $z_0 \in D$, and which is of unit modulus on the boundary ∂D of D. Thus give suitable constants c, d.

(b) Show that if $g(z)$ is analytic in a neighborhood of $\bar{D}$, has a simple zero at $z_0 \in D$ and no other zeros in D and is of unit modulus on the boundary ∂D, then $g(z)$ is in fact fractional linear. Hint: consider the quotient, $g(z)/f(z)$.

6. (a) Let $f(z) = z^2 - z + 1$. Compute the change in argument of $f(z)$ over

   (i) The positive real axis $\mathbf{R}_+$, $x \geq 0$ with $y = 0$.

   (ii) A large quarter circle $\{z : z = Re^{i\theta}, R \text{ fixed}, 0 \leq \theta \leq \pi/2\}$, in the limit, $R \to \infty$.

   (iii) The upper imaginary axis $y \geq 0$ with $x = 0$.

   (b) One can show that $|f(z)| \geq 3/4$ over the positive real axis and the upper imaginary axis. (You need not show this). Show that for $|\epsilon| < 3/4$,

$$g(z) \equiv z^2 - z + 1 + \frac{\epsilon}{z+1}$$

   has one zero in the first quadrant.

Complex Analysis General Exam, Winter 2003

1. Let $f(z) = 1/(1 - z^2)$. Obtain convergent Laurent series for f in the regions:

   a) $\{z : |z| < 1\}$.
   b) $\{z : |z| > 1\}$.
   c) $\{z : 0 < |z - 1| < 2\}$ (in powers of $(z - 1)$ and $(z - 1)^{-1}$).

2. Using contour integral techniques, compute:

   i)
$$\int_{\mathbf{R}} \frac{\cos x \, dx}{1 + x^2}$$

($\mathbf{R}$ is the real line.)

   ii)
$$\oint_{\{z:\,|z|=1\}} \frac{\cot z \, dz}{z^2}$$

3. Let $f(z) = z^{2003} - z + 2$.

   a) How many zeros does f have on the real line?

   b) How many zeros does f have in the unit disc $D = \{z : \ |z| < 1\}$?

   c) How many zeros does f have in the annulus $A = \{z : \ 1 \leq |z| < 2\}$?

4. Let $f(z)$, $z \in \mathbf{C}$, be defined by

$$f(z) \equiv \int_0^\infty e^{-zt - t^4} \, dt$$

   Show that $f(z)$ is analytic in $\mathbf{C}$.

5. a) Suppose that $f(z)$ is analytic in the unit disc $D = \{z : \ |z| < 1\}$, is continuous in $\bar{D}$, has no zeros in D, and is of constant modulus on the boundary of D, ∂D, i.e., $|f(z)| = 1$ for $z \in \partial D$. Use the maximum principle to show that $f(z)$ is constant.

The function $g(z) = (z - z_0)/(1 - \bar{z}_0 z)$, with $z_0 \in D$ has a simple zero at z_0 and is of unit modulus on the boundary ∂D. (You need not show this.)

b) Suppose that $h(z)$ is analytic in D, is continuous in $\bar{D}$, is of constant modulus on ∂D, but has a single simple zero at $z_0 \in D$. Show that

$$h(z) = \lambda g(z)$$

for some complex constant λ. (Consider $h(z)/g(z)$.)

COMPLEX ANALYSIS GENERAL EXAM, AUGUST, 2003

In each problem, justify all assertions, show calculations, and identify those theorems which you invoke in your arguments.

1. If f is analytic in a neighborhood of the origin and

$$\lim_{n\to\infty} n^n f(1/n) = 0,$$

prove that $f(z) \equiv 0$.

2. If $P(z)$ is a polynomial, and γ is the circle $|z-\alpha| = R$ show that

$$\frac{1}{2\pi i}\int_\gamma \overline{P(z)}dz = R^2\overline{P'(\alpha)}.$$

(Hint: expand $P(z)$ in powers of $z-\alpha$).

3. Evaluate, using contour integral techniques,

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}dx.$$

4. Suppose $f(z)$ is analytic in the unit disc, $D = \{z : |z| < 1\}$, with

$$f(1/2) = f(-1/2) = 0$$

and $|f(z| \leq 1$, for $z \in D$. Show that $|f(0)| \leq 1/4$. Hint: consider $f(z)/g(z)$, with the function

$$g(z) = (z^2 - 1/4)/(1 - z^2/4);$$

Note that $g(\pm 1/2) = 0$ and $g(z)$ is of unit modulus on ∂D, the boundary of D.

5. Prove that for $1 < a < \infty$, the function $z + a - e^z$ has only one zero in the left half-plane $\{\text{Re}z < 0\}$, and that this zero is on the real axis.

Complex Analysis General Exam — January 2004

1. How many zeros does the polynomial

$$P(z) = z^7 - 5z^4 + z^2 - 2$$

have inside the unit circle $|z| = 1$?

2. Let f be analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ with $\operatorname{Im} f(z) < 0$ there. Suppose $f(0) = -i$. Show

$$|f(z)| \leq \frac{1+|z|}{1-|z|} \quad \text{for all } z \text{ in } \mathbb{D}.$$

3. Let $\{f_n\}$ be a sequence of analytic functions on the unit disk $\mathbb{D}$. Suppose $f_n \to f$ uniformly on $A = \{\frac{1}{2} < |z| < 1\}$. Prove that there is an analytic g in $\mathbb{D}$ with $g = f$ on A.

4. Evaluate $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}\, dx$ by residue methods.

5. Suppose f is analytic in an open set containing $\{z : |z| \leq 3\}$, and let γ be the positively oriented circle $|z| = 3$. Assume $f \neq 0$ on γ, and

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}\, dz = 2,$$

while

$$\frac{1}{2\pi i}\int_\gamma \frac{z^2 f'(z)}{f(z)}\, dz = -4.$$

Find all the zeros of f in $\{|z| < 3\}$.

6. If $u + iv$ is analytic in an open set Ω, show uv is harmonic in Ω, but u^2 need not be harmonic in Ω.