Algebra General Exam - August 2022

Your UVa ID Number:

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. Let R denote the commutative ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

(a) (6 points) Construct two nonassociated factorizations of 6 into products of irreducible elements (you need to verify that all elements involved in these factorizations are indeed irreducible and the two factorizations are indeed nonassociated).

(b) (5 points) Using the factorizations from (a), construct an ideal $\mathfrak{a} \subset R$ which is **not** principal.

(c) (6 points) Show that the square $\mathfrak{a}^2 = \mathfrak{a}\mathfrak{a}$ of the ideal from part (b) is principal.

2. For a complex $n \times n$-matrix $A = (a_{ij})$ we let $\overline{A}$ denote the matrix $(\bar{a}_{ij})$ where the bar denotes the complex conjugation. Also, we let χ_A denote the characteristic polynomial of A.

(a) (4 points) If A and $\overline{A}$ are conjugate, prove that the characteristic polynomial χ_A has real coefficients.

(b) (7 points) Conversely, if A is diagonalizable and χ_A has real coefficients then A and $\overline{A}$ are conjugate.

(c) (5 points) Give an example of a nondiagonalizable complex matrix A such that χ_A has real coefficients but A and $\overline{A}$ are **not** conjugate.

3. (a) (7 points) Let G be a group, and $H \subset G$ be a normal subgroup. Assume that the center $Z(H)$ is $\{e\}$ and that every automorphism of H is inner. Show that G is the direct product $H \times C_G(H)$ where $C_G(H)$ is the centralizer of H in G. (*Hint.* Consider the action of G on H by inner automorphisms.)

(b) (8 points) Show that there is no group G such that the commutator subgroup $[G, G]$ is isomorphic to the symmetric group S_3. (You can assume without proof that S_3 satisfies the assumptions on H made in part (a).)

4. Let R be a commutative ring with identity. We say that a finitely generated R-module P is *projective* if P is a direct summand of a free R-module. That is, there is some $d \geq 1$ and another R-module Q such that $P \bigoplus Q \simeq R^d$ as R-modules.

(a) (6 points) Suppose P is a finitely generated projective R-module. Show that for every epimorphism of R-modules $\varphi\colon M \to N$, an arbitrary R-module homomorphism $\alpha\colon P \to N$ lifts to a homomorphism $\beta\colon P \to M$ such that $\varphi \circ \beta = \alpha$.

(b) (5 points) Show that if P_1 and P_2 are finitely generated projective modules then their tensor product $P_1 \otimes_R P_2$ is also a finitely generated projective R-module.

5. (a) (7 points) Let V and W be vector spaces over a field K, and let $v_1, \dots, v_n \in V$ be linearly independent vectors. Show that if $w_1, \dots, w_n \in W$ are such that $v_1 \otimes w_1 + \cdots + v_n \otimes w_n = 0$ in $V \otimes_K W$ then $w_1 = \cdots = w_n = 0$.

(b) (7 points) Again, let V and W be vector spaces over a field K, and let $x \in V \otimes_K W$. If $x = v_1 \otimes w_1 + \cdots + v_n \otimes w_n$ is a shortest presentation of x as a sum of simple tensors (i.e., x cannot be written as $v'_1 \otimes w'_1 + \cdots + v'_m \otimes w'_m$ with $m < n$) then the vectors $v_1, \dots, v_n$ (and likewise $w_1, \dots, w_n$) are linearly independent.
Note: Make sure you clearly state which properties of the tensor product you are using in both parts of this problem.

6. (8 points) Let α be a complex number satisfying $\alpha^6 + 3 = 0$. Show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension, and determine its Galois group. (*Hint.* The same is not true if 3 is replaced by 2.)

7. Assume that p is a prime number and $A \in GL_5(\mathbb{F}_p)$ is a matrix that satisfies $A^3 = 1$.

(a) (4 points) If $p \equiv 1 \mod 3$, show that A is diagonalizable.

(b) (7 points) Let $p = 11$. Classify all conjugacy classes of such matrices A.

8. (8 points) Let K be a subfield of $\mathbb{R}$, let $f(x) \in K[x]$ be an irreducible polynomial, and let L be the splitting field of f over K (i.e., the field obtained by adjoining to K all complex roots of f). Assume that the Galois group $\mathrm{Gal}(L/K)$ is abelian. Show that if one root of f is real then all roots are real.