Algebra general exam. January 11 2022, 9am -1pm

Your UVa ID Number:

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. (14 pts) Let p be a prime and $G = GL_2(\mathbb{F}_p)$, the group of invertible 2×2 matrices over $\mathbb{F}_p$.

(a) (6 pts) Find the order of G (with proof).

(b) (2 pts) Show that $U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in F \right\}$ is a Sylow p-subgroup of G

(c) (6 pts) Find the normalizer $N = N_G(U)$ and the number of Sylow p-subgroups of G (with proof).

Hint: In (c) one can solve either part of the problem first and then use the answer to solve the other part.

2. (14 pts) Show that there exist precisely 3 isomorphism classes of groups G that contain a subgroup H of index 2 which is infinite cyclic (i.e., is isomorphic to $\mathbb{Z}$).

Hint: Consider separately the cases where G is abelian and G is non-abelian. In the non-abelian case consider a natural action of G on H and use it to show that G must have an element of order 2.

3. (12 pts) Let $f(x, y), g(x, y) \in \mathbb{C}[x, y]$ be two polynomials that do not have a (non-constant) common factor.

(a) (7 pts) Show that f and g are relatively prime as elements of $\mathbb{C}(x)[y]$ and $\mathbb{C}(y)[x]$ (where $\mathbb{C}(x)$ and $\mathbb{C}(y)$ are the fields of rational functions, i.e. the fraction fields of $\mathbb{C}[x]$ and $\mathbb{C}[y]$, respectively). Give a detailed argument.

(b) (5 pts) Show that the system of polynomial equations $f(x, y) = 0$ and $g(x, y) = 0$ has finitely many solutions in $\mathbb{C}^2$. **Hint:** Use (1) and the fact that $\mathbb{C}(x)[y]$ and $\mathbb{C}(y)[x]$ are PIDs (make sure to explain why the latter is true).

4. (12 pts) Let $R = \mathbb{Z}[\sqrt{-11}] = \{a + b\sqrt{-11} : a, b \in \mathbb{Z}\} \subset \mathbb{C}$, and let $I = (3, 1 + \sqrt{-11})$ be the ideal of R generated by 3 and $1 + \sqrt{-11}$.

(a) (6 pts) Prove that I is maximal.

(b) (6 pts) Prove that I is not principal.

5. (10 pts) Let V be a finite-dimensional vector space over an arbitrary field F (not necessarily algebraically closed!) and $T : V \to V$ an F-linear map. Prove that the following two conditions on T are equivalent:

(1) The characteristic polynomial and the minimal polynomial of T coincide
(2) There exists $v \in V$ such that V is spanned by the set $\{v, T(v), T^2(v), \ldots\}$

Hint: Consider V as an $F[x]$-module with x acting as T and use a suitable structure theorem for such modules.

6. (14 pts) In each part determine whether the statement is TRUE (in all cases) or FALSE (in at least one case) and prove your claim. An answer (correct or incorrect) without explanation will not receive any credit.

(a) (3 pts) Let R be a commutative domain with 1 and M an R-module. If $x \in M$ and $y \in M$ are both torsion elements, then $x + y$ is also a torsion element.
(b) (3 pts) If K and L are fields, then $K \otimes_{\mathbb{Z}} L$ is nonzero.
(c) (4 pts) If R is a commutative ring with 1 and every R-module is free, then R is a field.
(d) (4 pts) If R is a commutative ring with 1 and M is a finitely generated R-module, then every submodule of M is finitely generated.

7. (10 pts) Let p be a prime and let F be a field of order p^6. Let

$$Prim(F) = \{\alpha \in F : \mathbb{F}_p(\alpha) = F\}.$$

(a) (5 pts) Find (with proof) an explicit formula for $|Prim(F)|$.
(b) (5 pts) Let $Irr_6(p)$ denote the set of all monic irreducible polynomials of degree 6 over $\mathbb{F}_p$. Find (with proof) a simple relation between $|Irr_6(p)|$ and $|Prim(F)|$. **Note:** You are not allowed to use the general formula for the number of irreducible polynomials of a given degree over $\mathbb{F}_p$.

8. (14 pts) Let F be a field, and let $f(x) \in F[x]$ be a separable irreducible polynomial of degree n.

(a) (5 pts) Let $\alpha \neq \beta$ be distinct roots of F (in some fixed field extension K of F). Prove that $[F(\alpha, \beta) : F] \leq n(n-1)$.
(b) (5 pts) Let α and β be as in (a). Prove that

$$[F(\alpha + \beta) : F] \leq \binom{n}{2} = \frac{n(n-1)}{2}.$$

Hint: Use the action of a suitable Galois group.
(c) (4 pts) Assume that $F = \mathbb{Q}$ and n is prime. Give an explicit example of f and α and β satisfying the above conditions such that the equality in (a) holds (you are NOT allowed to choose your prime n). Prove that your example has the required properties.