Algebra general exam. August 17 2021, 9am -1pm

Your UVa ID Number:

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. (12 pts) Let $n \in \mathbb{N}$ and let $A = (a_{ij}) \in Mat_n(\mathbb{Q})$ be the matrix given by $a_{ij} = i$ for $1 \le i, j \le n$, that is,

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 2 & 2 & \dots & 2 \\ \dots & \dots & \dots & \dots \\ n & n & \dots & n \end{pmatrix}$$

Compute

(a) The characteristic polynomial of A
(b) The minimal polynomial of A
(c) The Jordan canonical form of A
(d) The rational canonical form of A

2. Given a group G, denote by $\mathrm{Aut}(G)$ its group of automorphisms.

(a) (7 pts) Let $G = \mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^b}$ where $a \ge b > 0$. Show that $\mathrm{Aut}(G)$ is non-abelian by explicitly constructing two noncommuting automorphisms. **Hint:** It may be helpful to start with the case $a = b = 1$.

(b) (8 pts) Let G be a finite abelian group. Show that $\mathrm{Aut}(G)$ is abelian $\iff$ G is cyclic. **Hint:** Use (a). Remember that you are allowed to do this even if you did not solve (a).

3.

(a) (8 pts) Let p be a prime and G a group or order p^3. Prove that any two elements x, y of G which are conjugate in G commute (that is, $xy = yx$). **Hint:** Prove that any element g of G is contained in some abelian normal subgroup of G (which depends on g).

(b) (7 pts) Give, with arguments, an example of a group G of order 16 and two elements x, y of G which are conjugate in G but do *not* commute.

4. (14 pts) Let R be a (commutative) UFD (with 1), let $f \in R$, and assume that f is non-unit and nonzero. Let $R_f = R[\frac{1}{f}]$ (you can think of R as the ring of fractions of R with the set of denominators $D = \{1, f, f^2, \ldots\}$ or as the subring of the field of fractions $Frac(R)$ generated by R and $\frac{1}{f}$). Prove that

$$R_f^\times \cong R^\times \times \mathbb{Z}^m \text{ for some } m \in \mathbb{N}$$

(Here $S^\times$ is the multiplicative group of S). Give a detailed argument.

5. Let R be a commutative ring with 1 and let M and N be finitely generated R-modules.

(a) (5 pts) Prove that $M \otimes_R N$ is finitely generated.

(b) (9 pts) Assume in addition that M is Noetherian. Prove that $M \otimes_R N$ is Noetherian. **Note:** You can use standard properties of Noetherian modules (unless they are equivalent or almost equivalent to the statement of (b)), but state clearly what you are using.

6. Consider $f(x) = x^4 + ax^2 - 1 \in \mathbb{Q}[x]$ where $a \in \mathbb{Z} \setminus \{0\}$. Let K be the splitting field of $f(x)$ over $\mathbb{Q}$.

(a) (5 pts) Show that $f(x)$ is irreducible

(b) (4 pts) Show that $i \in K$

(c) (7 pts) Determine the isomorphism class of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ (your answer should be of the form $\mathrm{Gal}(K/\mathbb{Q}) \cong G$ where G is a familiar group).

7.

(a) (7 pts) Let $n \in \mathbb{N}$, let F be an algebraically closed field either of characteristic 0 or of characteristic $p > 0$ where p does not divide n. Prove that F contains a primitive n^{th} root of unity.

(b) (7 pts) According to (a) $\overline{\mathbb{F}}_3$ (the algebraic closure of a field with 3 elements) contains a primitive 13^{th} root of unity, call it ω. Compute $[\mathbb{F}_3(\omega) : \mathbb{F}_3]$ (with proof).