Algebra general exam. January 25 2021, 9am -1pm

Your UVa ID Number:

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. Let F be a field. Let $f(x) \in F[x]$ be an *irreducible separable* polynomial of degree n, let G be the Galois group of f over F, and assume that G is abelian.

(a) (7 pts) Prove that if $g \in G$ is any non-trivial element, then g does not fix any of the roots of f.

(b) (3 pts) Now assume that $F \subseteq \mathbb{R}$ and n is odd. Prove that all roots of f must be real.

(c) (4 pts) Now let $F = \mathbb{Q}$. Prove that there are infinitely many n for which there exists f as above with a non-real root and cyclic G.

2. (12 pts) Classify conjugacy classes of matrices $A \in \mathrm{GL}_7(\mathbb{Q})$ such that $A^3 = -Id$.

3. (12 pts) Let R be a commutative ring with 1, and assume that $|R| > 1$. Prove that R has at least one minimal prime ideal. Make sure to include all the details. **Hint:** Use Zorn's lemma "backwards".

Note: A prime ideal P of R is called a *minimal prime ideal* if R has no prime ideals strictly contained in P. In particular, the zero ideal is a minimal prime whenever it is prime.

4. (12 pts) Let G be a finite abelian group and let p be a prime. Prove that the number of elements of order p in G is equal to the number of nontrivial homomorphisms from G to $\mathbb{Z}_p$.

Hint: Calculate both numbers separately.

5. Let $n \geq 2$ be an integer, let $g = (1, 2, 3, \ldots, n) \in S_n$ and $H = \langle g \rangle$.

(a) (6 pts) Prove that the centralizer of g in S_n is equal to H.

(b) (7 pts) Now assume that n is prime, and let N be the normalizer of H in S_n. Prove that $|N| = n(n-1)$.

6. In both parts of this problem, R is a commutative ring with 1, assume that $|R| > 1$, and M is a flat R-module, that is, assume that

Whenever we have an injective homomorphism $N \xrightarrow{f} N'$ of R-modules, the induced map $f \otimes 1_M : N \otimes_R M \to N' \otimes_R M$ is also injective.

(1) (4 pts) Assume that R is a domain. Prove that M must be torsion-free.

(2) (8 pts) Now let R be arbitrary. Prove that the following are equivalent:

  (a) for every nonzero R-module $N \neq 0$ we have $N \otimes_R M \neq 0$

  (b) for every maximal ideal $\mathfrak{m}$ of R we have $\mathfrak{m}M \neq M$.

Note: You may use without proof standard isomorphisms of the form $(R/I) \otimes_R M \cong \ldots$ where I is an ideal of R.

7. (13 pts) Recall that a field F is called *perfect* if either F has characteristic zero or F has characteristic $p > 0$ and every element of F is equal to a^p for some $a \in F$. Prove that F admits a finite inseparable extension if and only if F is not perfect.

8. (12 pts) Let $p \neq 3$ be a prime and $R = \mathbb{F}_p[x]/(x^3 - 1)$. Describe the multiplicative group $R^\times$ as a direct product of cyclic groups.

Note: Your answer can (and should) involve cases, but should be expressed explicitly in terms of p.