Algebra general exam. August 19, 2020, 9am -1pm

Your UVa ID Number:

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. (12 pts) Let A be a finitely generated abelian group. Prove that the following are equivalent:

(a) $\dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}) \leq 1$
(b) $\mathrm{Aut}(A)$ is finite.

Hint: Use classification of finitely generated abelian groups. What does the condition $\dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}) \leq 1$ tell you about a standard decomposition of A?

2. Let $n \geq 3$ be an integer. Denote by S_n the symmetric group on $\{1, \ldots, n\}$ and by D_{2n} the dihedral group of order $2n$.

(a) (7 pts) Prove that for every $k \geq 3$, there exists an injective homomorphism $\varphi : D_{2k} \to S_k$ whose image contains a k-cycle.
(b) (7 pts) Prove (using (a) or otherwise) that every element of S_n can be written as a product of two elements of order ≤ 2.

3. (10 pts) Let $R = \mathbb{Z}[\sqrt{3}]$. You may assume without proof that R is a Euclidean domain. Let $p \in \mathbb{N}$ be a prime number with $p \equiv 5 \mod 12$. Prove that p is a prime element of R. **Hint:** $12 = 3 \cdot 4$.

4. Let R be a commutative ring with 1. An ideal I of R is called **irreducible** if I cannot be written as $I = J \cap K$ where J and K are both ideals strictly containing I.

(a) (4 pts) Prove that every prime ideal is irreducible.

(b) (8 pts) Assume that R contains a nonzero nilpotent element (that is, a nonzero element a such that $a^n = 0$ for some n). Prove that R contains an irreducible ideal which is not prime.

Hint: Fix a nonzero nilpotent element $a \in R$. Consider the set of all ideals NOT containing a and show that this set has a maximal element.

5. Let F be an algebraically closed field of char $F \neq 2$.

(a) (7 pts) Let J be a Jordan block of size n with eigenvalue λ over F. Determine the Jordan canonical form of the matrix J^2. **Hint:** Consider the cases $\lambda = 0$ and $\lambda \neq 0$ separately.

(b) (7 pts) Let $A \in Mat_5(F)$. Determine necessary and sufficient conditions for A to have a square root, i.e. for there to exist a matrix $B \in Mat_5(F)$ such that $A = B^2$. State your answer in the form: A has a square root $\iff$ $JCF(A)$ satisfies certain conditions. Make sure to prove your answer.

6. Let K be a field of characteristic 0, and fix some algebraic closure $\overline{K}$ of K. Let L/K be a finite extension, with $L \subseteq \overline{K}$, and let α be an element of $\overline{K}$.

(a) (7 pts) Prove that there is a surjective K-algebra homomorphism $\rho : L \otimes_K K(\alpha) \twoheadrightarrow L(\alpha)$.

(b) (7 pts) Prove that there exists an injective K-algebra homomorphism (not necessarily sending 1 to 1) $\iota : L(\alpha) \to L \otimes_K K(\alpha)$.

Hint: $K(\alpha) = \dfrac{K[x]}{(\mu_{\alpha,K}(x))}$.

7. Let p be an odd prime number and set $q = p^2$. Consider the finite field $\mathbb{F}_q$.

(a) (6 pts) Show that $\mathbb{F}_q$ contains a primitive 8-th root of unity w and that the element $\alpha = w + w^{-1}$ satisfies $\alpha^2 = 2$.

(b) (6 pts) Show that $\alpha \in \mathbb{F}_p$ if and only if $p \equiv 1$ or $7 \bmod 8$. **Hint:** Check when is $\alpha^p = \alpha$.

8. Let K be the splitting field of the polynomial $f(x) = (x^3 - 11)(x^2 + x - 1)$ over $\mathbb{Q}$.

(a) (6 pts) Find (with proof) the degree of the extension $K/\mathbb{Q}$.

(b) (6 pts) Determine the isomorphism class of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ and prove your answer. State your answer in the form $\mathrm{Gal}(K/\mathbb{Q}) \cong G$ where G is a "familiar" group, e.g. $\mathrm{GL}_2(\mathbb{F}_2)$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$.