Algebra general exam. January 8, 2020, 9am -1pm

Your UVa ID Number:

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1.

(a) (7 pts) Let C be a cyclic group of order $n \geq 2$. Explain briefly why $\mathrm{Aut}(C) \cong \mathbb{Z}_n^\times$ (the multiplicative group of $\mathbb{Z}_n$) and why $\mathrm{Aut}(C) \cong \mathbb{Z}_{n-1}$ if n is prime.

(b) (7 pts) Let G be a finite group, let p be the smallest prime dividing $|G|$, and suppose that G contains a normal subgroup C of order p. Prove that C lies in the center of G.

2. (10 pts) Let A be the subgroup of $\mathbb{Z}^2$ generated by $(2, 6)$ and $(4, 8)$. Prove that $A \cong \mathbb{Z}^2$ and find (with proof) elements $v_1, v_2 \in \mathbb{Z}^2$ and $n_1, n_2 \in \mathbb{N}$ such that $\mathbb{Z}^2 = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ and $A = \mathbb{Z}(n_1v_1) \oplus \mathbb{Z}(n_2v_2)$.

3. (10 pts) Let $R = \mathbb{Z}[i]$, the ring of Gaussian integers. Find (with complete proof!) the number of ideals of R which contain 30.

4. Let R be a commutative ring with 1 and let $x \in R$.

(a) (3 pts) Prove that $x \notin R^\times$ if and only if $x \in M$ for some maximal ideal M of R.

(b) (9 pts) Prove that the following are equivalent:

(i) $x \in M$ for every maximal ideal M of R

(ii) $1 + xy \in R^\times$ for every $y \in R$

5. (12 pts) Let F be a field and let $A, B \in Mat_n(F)$ for some $n \in \mathbb{N}$. Suppose that $A^2 = B^2 = I$ and $\mathrm{rk}(A - I) = \mathrm{rk}(B - I)$. Prove that A and B are similar. **Hint:** Consider separately the cases $\mathrm{char}\, F \neq 2$ and $\mathrm{char}\, F = 2$.

6. In each part of this problem determine if the given objects are isomorphic:

(a) (4 pts) $\mathbb{R}$ and $\mathbb{C}$ as $\mathbb{Q}$-vector spaces
(b) (3 pts) $\mathbb{C}$ and $\mathbb{R} \times \mathbb{R}$ as rings
(c) (4 pts) $\mathbb{R}[x]/(x^2 - 1)$ and $\mathbb{R} \times \mathbb{R}$ as rings
(d) (4 pts) $\mathbb{R}[x]/(x - 1)$ and $\mathbb{R}[x]/(x + 1)$ as $\mathbb{R}[x]$-modules

7. Let F, K_1, K_2 and L be fields with $F \subseteq K_1 \subseteq L$ and $F \subseteq K_2 \subseteq L$. Recall that the tensor product $K_1 \otimes_F K_2$ has a natural structure of a ring where multiplication of simple tensors is given by $(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$ for all $a, c \in K_1$ and $b, d \in K_2$.

(a) (5 pts) Prove that there exists a unique **ring homomorphism** $\pi : K_1 \otimes_F K_2 \to L$ such that $\pi(a \otimes b) = ab$ for all $a \in K_1$, $b \in K_2$.
(b) (2 pts) Assume that $K_1 \otimes_F K_2$ is a field. Prove that π is injective.
(c) (5 pts) Assume that $K_1 \cap K_2 \neq F$. Prove that π is not injective.

8. Let K/F be a field extension. Suppose that $K = F(a, b)$ for some $a, b \in K$ such that $a^2 \in F$ and $b^2 \in F$.

(a) (3 pts) Prove that $[K : F] \leq 4$
(b) (4 pts) Give a specific example (with full proof) where $[K : F] = 4$
(c) (4 pts) Assume that $\mathrm{char}\,(F) \neq 2$. Prove that the extension K/F is Galois
(d) (4 pts) Now assume that F is finite. Prove that $[K : F] \leq 2$.