Algebra general exam. August 19, 2019, 9am -1pm

Your UVa ID Number:

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1.

(a) (6 pts) Let G be a group of order 12. Prove that G has a normal Sylow subgroup.

(b) (9 pts) Prove that there are at least 5 pairwise non-isomorphic groups of order 12 (in fact, 5 is the exact number of isomorphism classes, but you are not asked to prove this). Partial credit for exhibiting fewer than 5 non-isomorphic groups (with proof) will be given.

2. (10 pts) Let $n \geq 3$ be an integer and let S_n be the symmetric group on $\{1, 2, \dots, n\}$). Let H be a subgroup of S_n with $[S_n : H] = n$. Prove that

$$H \cong S_{n-1}.$$

Hint: Start by constructing a suitable action of S_n associated to H. You may use the description of normal subgroups of S_n without proof.

3. (10 pts) Let m and n be positive integers with $m \mid n$, and let $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the natural projection. Proved that the associated map of the groups of units $f : (\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/m\mathbb{Z})^\times$ is surjective.

4. Let R be a commutative ring with 1, let S be a subring of R with 1, and let M and N be R-modules

(a) (5 pts) Prove that there exists a surjective S-module homomorphism $\varphi : M \otimes_S N \to M \otimes_R N$ such that $\varphi(m \otimes_S n) = m \otimes_R n$ for all $m \in M$ and $n \in N$. Also prove that such φ is unique.

(b) (6 pts) Assume that R and S are both fields, $R \neq S$ and M and N are both nonzero. Prove that φ in part (a) is not injective. **Warning:** M and N are not assumed to be finitely generated.

(c) (4 pts) Suppose that we only know that R is a field. Does the conclusion of (b) remain true?

5. Let F be a field, let V be a vector space over F of finite dimension n, and let $T : V \to V$ be an F-linear map.

(a) (5 pts) Assume that F is algebraically closed. Prove that V has at least $n + 1$ T-invariant subspaces (including 0 and V)

(b) (2 pts) Give an example showing that if F is not algebraically closed, the conclusion of (a) may be false.

(c) (7 pts) Now assume that T is diagonalizable over F and has n distinct eigenvalues. Prove that the number of T-invariant subspaces depends only on n and find that number.

6. Let $R = \mathbb{Z}[x]$.

(a) (6 pts) Let I be an ideal of R which contains a MONIC polynomial of degree n, call it $p(x)$. Prove that I can be generated (as an ideal) by at most $n + 1$ elements. **Hint:** Consider the quotient $I/(p(x))$.

(b) (6 pts) Now let $M = (2, x)$ and $I = M^2 = (4, 2x, x^2)$. Prove that I cannot be generated (as an ideal) by less than 3 elements. **Hint:** Consider the quotient M^2/M^3.

7. Let p and q be distinct primes, let $F = \mathbb{Q}(\sqrt[3]{p}, \sqrt[5]{q})$ and let K be the Galois closure of F over $\mathbb{Q}$.

(a) (3 pts) Prove that $[F : \mathbb{Q}] = 15$.

(b) (4 pts) Prove that $[K : \mathbb{Q}] = 120$.

(c) (4 pts) Prove that $\mathrm{Gal}(K/\mathbb{Q})$ has a normal subgroup of order 15.

(d) (3 pts) Prove that $\mathrm{Gal}(K/\mathbb{Q})$ has no normal subgroup of order 8.

8. (10 pts) Let q be a prime power, let $\mathbb{F}_q$ be a field of order q, and let $a \in \mathbb{F}_q$ be a nonzero element. Consider the polynomial

$$f(x) = x^{q+1} - a.$$

Let L be the splitting field of $f(X)$ over $K = \mathbb{F}_q$. Prove that $[L : K] = 2$. **Note:** Make sure to prove that the degree is equal to 2, not just ≤ 2.

Hint: Let α be a root of f. What can you say about the (multiplicative) order of α?