Algebra general exam. January 11, 2019, 9am -1pm

Your UVa ID Number:

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. Let X and Y be non-abelian simple groups.

(a) (7 pts) Let $G = X \times Y$. Prove that the only normal subgroups of G are G, $X \times \{1\}$, $\{1\} \times Y$ and the trivial subgroup.

Hint: Show that if N is a normal subgroup of G not contained in $X \times \{1\}$ (respectively, $\{1\} \times Y$), then N contains an element of the form $(1, y)$ with $y \neq 1$ (respectively, $(x, 1)$ with $x \neq 1$).

(b) (7 pts) Use (a) to prove that $\mathrm{Aut}(X \times X)$ is isomorphic to a semi-direct product of $\mathrm{Aut}(X) \times \mathrm{Aut}(X)$ and $\mathbb{Z}_2$ (a cyclic group of order 2).

2. Let G be a finite group of order n.

(a) (7 pts) Prove that there exists an injective homomorphism $\varphi : G \to S_n$ such that for every $g \in G$, the permutation $\varphi(g)$ is a product of n/k disjoint cycles of length k (for some k depending on g)

(b) (6 pts) Now assume that n is even and a Sylow 2-subgroup of G is cyclic. Use (a) to prove that G has a subgroup of index 2.

3. (10 pts) Let $R = \mathbb{Z}[x]$. Find the number of maximal ideals of R which contain $x^2 + 1$ and 15 and find explicit generators for each such ideal. **Hint:** Reduce to a question about Gaussian integers.

4. Let F be a field with $\mathrm{char}(F) \neq 2$, let V be a finite-dimensional vector space over F, and let B be a symmetric bilinear form on V.

(a) (4 pts) Prove that if $B \neq 0$, there exists $v \in V$ such that $B(v, v) \neq 0$.

(b) (4 pts) Prove that for any $v \in V$ with $B(v, v) \neq 0$ there exists a subspace W such that $V = Fv \oplus W$ and $W \perp v$, that is, $B(w, v) = 0$ for all $w \in W$.

(c) (4 pts) Use (a) and (b) to prove that there is a basis $\{v_n\}$ of V such that $B(v_i, v_j) = 0$ for all $i \neq j$.

5. Let F be an algebraically closed field, $n \in \mathbb{N}$ and $A \in \mathrm{GL}_n(F)$ an invertible $n \times n$ matrix over F.

(a) (9 pts) Assume that $\mathrm{char}(F) \neq 2$. Prove that if A^2 is diagonalizable, then A is also diagonalizable over F

(b) (4 pts) Give an example where $\mathrm{char}(F) = 2$, A^2 is diagonalizable, but A is not diagonalizable.

6. Let R be a commutative ring with 1, let M be an R-module and N a submodule of R.

(a) (8 pts) Prove that if N and M/N are both finitely generated, then M is finitely generated

(b) (5 pts) Give an example where M is finitely generated and N is not.

7. Let $F = \mathbb{Q}(\sqrt[6]{3}, i)$.

(a) (4 pts) Prove that $[F : \mathbb{Q}] = 12$

(b) (4 pts) Prove that the extension $F/\mathbb{Q}$ is Galois

(c) (4 pts) Prove that the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to D_{12}, the dihedral group of order 12.

8. Let $p > 2$ be a prime, let $\mathbb{F}_p$ be a field of order p and let $\overline{\mathbb{F}_p}$ be an algebraic closure of $\mathbb{F}_p$. Let $f(x) = x^m + 1$ for some $m \in \mathbb{N}$. Assume that f is irreducible, and let α be a root of f in $\overline{\mathbb{F}_p}$.

(a) (5 pts) Prove that the multiplicative order of α is equal to $2m$.

(b) (5 pts) Prove that $2m$ divides $p^m - 1$ and $2m$ does not divide $p^k - 1$ for any $0 < k < m$.

(c) (3 pts) Prove that $m \neq 4$.