ALGEBRA GENERAL EXAM

August 20, 2018

Your UVa ID Number:

- Please show all your work and justify any statements that you make.
- State any theorem you use clearly and fully.
- Vague statements and unclear arguments will not be accepted as progress towards a solution.
- You may assume the statement of an earlier question proven in order to solve a later one.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

(1) Let G be a group, and let $\gamma_i(G)$ denote the i^{th} term of the lower central series of G. That is, $\gamma_1(G) = G$ and inductively $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. Let $\mathrm{IAut}(G) < \mathrm{Aut}(G)$ be the group of automorphisms of G which induce the trivial automorphism of $G/[G,G]$. Prove the following statements.

(a) (4 points) For each $i \geq 1$, we have $\gamma_i(G) < G$ is characteristic and $\gamma_i(G)/\gamma_{i+1}(G)$ is abelian.

(b) (6 points) If $\phi \in \mathrm{IAut}(G)$ and $i \geq 1$, then ϕ induces the trivial automorphism of $\gamma_i(G)/\gamma_{i+1}(G)$. You may use without proof the fact that for any group G, there is an inclusion

$$[[G,G], \gamma_i(G)] \subset \gamma_{i+2}(G).$$

(2) (10 points) Let G be a non–cyclic group of order 57. Determine the number of elements of all orders of G.

(3) Let R be a commutative ring such that all prime ideals are finitely generated. Prove that R is Noetherian by completing the following steps.

(a) (5 points) Let X be the set of non-finitely generated ideals of R. Prove that if R is not Noetherian then X has a maximal element, say I.

(b) (1 point) Prove that there exist elements $x, y \in R$ such that $x, y \notin I$ but such that $xy \in I$.

(c) (3 points) Prove that $I_x = I + Rx$ and $J_x = \{r \in R \mid rx \in I\}$ are finitely generated ideals. Here, x is the same element as in the previous part.

   (d) (6 points) Conclude that I is finitely generated and derive a contradiction.

(4) Let R be an integral domain and let M be a nontrivial torsion R–module.
   (a) (5 points) If M is finitely generated then the annihilator of M in R is nontrivial. Recall that the annihilator of M is the ideal $\{r \mid rm = 0 \text{ for all } m \in M\}$.
   (b) (5 points) Find an integral domain R and a torsion module M over R whose annihilator is the zero ideal.

(5) Let F be a field and let n be a natural number.
   (a) (10 points) Let $F = \mathbb{R}$. Classify the matrices $A \in M_n(\mathbb{R})$ satisfying $A^3 = A$, up to similarity. That is to say, exhibit a matrix in each similarity class satisfying $A^3 = A$.
   (b) (5 points) For an appropriate F and n, find a matrix $A \in M_n(F)$ which is not diagonalizable and which satisfies $A^3 = A$.

(6) (13 points) Let $p(x)$ be a polynomial defined over $\mathbb{R}$, and suppose that p takes on rational values at rational numbers. Prove that the coefficients of p are rational (Hint: use the Vandermonde determinant). Does the statement remain true if the rationals are replaced by the integers?

(7) (14 points) What is the Galois group of $x^5 - 1$ over a field with 7 elements?

(8) (13 points) Describe the splitting field of $x^4 + x^2 + 1$ over $\mathbb{Q}$. That is, describe elements of the algebraic closure of $\mathbb{Q}$ which need to be adjoined in order to obtain the splitting field. What is the degree of the splitting field over $\mathbb{Q}$?