Algebra general exam

January 11, 2018

Your name:

- Please show all your work and justify any statements that you make.
- State any theorem you use clearly and fully.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement of an earlier question proven in order to solve a later one.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. (10 points) Classify, up to isomorphism, all finite groups of order $2p$, where p is a prime number.

2. (12 points, 6 points each) Consider the ring
   $R = \mathbb{Z}[\sqrt{-11}] = \{m + n\sqrt{-11} \mid m, n \in \mathbb{Z}\}$.
   (a) Is R a UFD? Give arguments for your answer.
   (b) Exhibit an ideal I in R which is not principal. *Show* that your I is not principal.

3. (15 points, 5 points each) Decide in each of the following three cases whether the given polynomial is irreducible. Include arguments.
   (a) $x^2 - 2i$ in $\mathbb{Z}[i][x]$;
   (b) $x^3 - 49x^2 + (3 + \sqrt{2})x + 7$ in $\mathbb{Z}[\sqrt{2}][x]$;
   (c) $x^2 + xy + y^2$ in $\mathbb{C}[x, y]$.

4. (12 points) Let A be a finite abelian group of order n, p a prime divisor of n and $n = p^k m$ with $k, m \in \mathbb{N}$ such that $(p, m) = 1$. Denote by A_p the Sylow p-subgroup of A.
   (a) (8 points) Show that the abelian groups A_p and $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ are isomorphic.
   (b) (4 points) Describe $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} A$ as an abelian group without using tensor products but (certain) invariants of A.

5. (16 points) We set $M := M_3(\mathbb{Q})$ and denote by 0 the zero matrix and by I the identity matrix of M.
(a) (2 points) Prove or disprove: If $A \in M$ satisfies $A^6 = 0$, then also $A^3 = 0$.
(b) (4 points) Classify, up to similarity, all matrices in M satisfying $A^6 = 0$. Exhibit one representative for each such similarity class.
(c) (2 points) Prove or disprove: If $A \in M$ satisfies $A^6 = I$, then also $A^3 = I$.
(d) (8 points) Classify, up to similarity, all matrices in M satisfying $A^6 = I$. Exhibit one representative for each such similarity class.

6. (10 points) Let $n \geq 2$ be a natural number, F a field and $A = (a_{ij}) \in M_n(F)$ the matrix with entries $a_{ij} = j \cdot 1_F \in F$ for all $1 \leq i, j \leq n$.
Determine the characteristic polynomial, the minimal polynomial and the JCF of A.
Hint: The result may depend on the characteristic of F.

7. (15 points)
(a) (8 points) Construct, using cyclotomic fields, a Galois extension K of $\mathbb{Q}$ of degree 3. Include arguments.
(b) (7 points) Find, explicitly, a polynomial $f(x) \in \mathbb{Q}[x]$ such that your field K in (a) is the splitting field of $f(x)$ over $\mathbb{Q}$.

8. (10 points) Let $M \mid K \mid F$ be a tower of finite field extensions such that $M \mid F$ and $K \mid F$ are both Galois. Assume that the Galois group $G(M|K)$ is cyclic. Prove that $L \mid F$ is Galois for every intermediate field L with $M \mid L \mid K$.